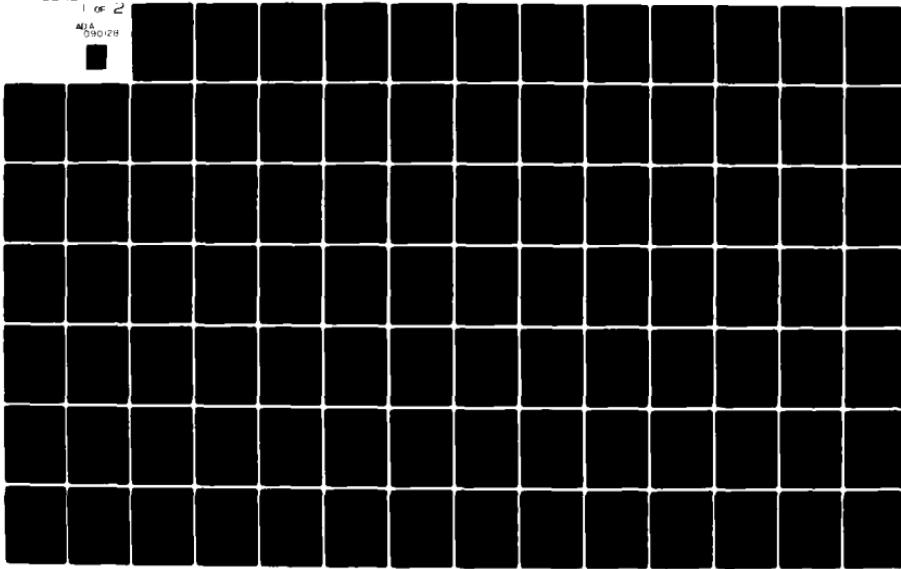


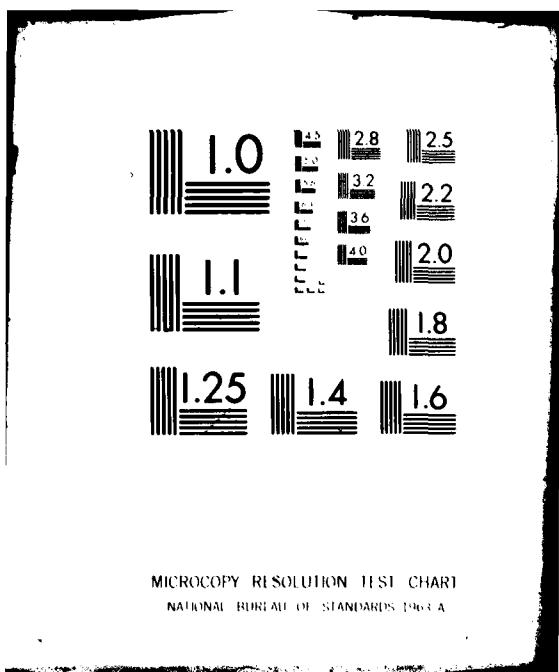
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UNIVERSITY OF RHODE ISLAND  
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ON  
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APPLICATIONS OF EFFICIENT SUBSET SELECTION  
TO DIGITAL FILTERING AND TO SIGNAL RESOLUTION

JAFIR KHORAMMI  
AND  
DONALD TUFTS

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**ABSTRACT**

The subset selection algorithm is extended to search for a best subset from a large set of complex-valued basis functions. This algorithm is used to design digital finite-duration impulse response (FIR) filters having fewer coefficients than conventional FIR filters. An optimum conventional FIR filter is derived which has best uniform spacing of the fixed number of samples which are to be used, and examples are presented which show that, for the same number of coefficients, the complex-subset-selection filter can give better results than the optimum conventional filter.

The complex subset selection method is also applied to estimation of the frequencies of sinusoids in the presence of noise. A windowing technique is introduced to increase the efficiency and accuracy of the algorithm for frequency estimates. The results are compared with Cramer-Rao bounds.

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## CHAPTER 1

### INTRODUCTION

This is a report of work done on digital filter design and sinusoidal-tone estimation by the complex subset selection method. In the complex subset selection method, a set of complex basis functions are chosen and a best subset is selected to represent a desired function. The desired functions investigated are an ideal transfer function of a low-pass digital filter, and the sum of two sinusoidal tones. The summed squared-magnitude error is used as the criterion for subset selection.

One can design a digital lowpass FIR filter by use of the window method (1), frequency sampling (2), and other techniques (3), (4). In all of these methods, the input data elements are uniformly spaced in time with respect to one another. Instead, we search for the best M data elements, which may or may not be uniformly located, to minimize the least squares error in approximating the desired transfer function. We compare the result with a filter having the same number of coefficients, but uniformly spaced locations.

We will also investigate sinusoidal-tone parameter estimation, which is a very important subject in communication systems. Linear prediction (5) and spectral estimation (6) are two well known methods for estimating

parameters of tones, such as frequencies. Both of these methods can produce biased estimates of the signal parameters when more than one signal is present (7). We use complex subset selection to estimate frequencies of tones without such bias. The effect of additive noise is discussed, and the results are compared with Cramer-Rao bounds computed by Rife (8). A windowing technique is applied to increase the accuracy and the efficiency of the algorithm. By "windowing technique", we mean that a small set of basis functions is always chosen over potentially productive intervals, which will be searched for a best subset, and other intervals are ignored. That is, potentially productive intervals are first located, and only these intervals are searched for the desired frequencies.

The basis for complex subset selection is reviewed in the following paragraphs:

Regression analysis may be broadly defined as the analysis of relationships among variables. It is one of the most widely used statistical tools because it provides a simple method for establishing a linear functional relationship among variables. The relationship is expressed in the form of an equation connecting the response or dependent variable  $\hat{Y}$ , and one or more independent variables,  $x_1, x_2, \dots, x_N$ . The regression equation takes the form

$$\hat{y} = b_1 x_1 + b_2 x_2 + \dots + b_N x_N \quad (1.1)$$

where  $b_1, b_2, \dots, b_N$  are called the regression coefficients and are determined from the data. However, in some applications, one needs to select  $M$  out of the  $N$  ( $M < N$ ) independent variables to best approximate  $y$  by

$$\hat{y} = a_1 u_1 + a_2 u_2 + \dots + a_M u_M \quad (1.2)$$

where  $U = \{u_1, \dots, u_M\}$  is a subset selected from  $X = \{x_1, \dots, x_N\}$ . Examining all  $\binom{N}{M} = N!/(N-M)!M!$  different subsets to select the best subset involves a lot of computation, especially when  $N$  is large.

Hocking and Leslie (H-L) (10) have developed an algorithm for finding the best subset. In their algorithm, they first test the variables which are superior. That is, the ones which, when omitted from the linear regression equation, result in a large residual error. If a best subset is obtained, they stop. Otherwise, the variables with smaller residual error are tested until a best subset is obtained. They have developed a criterion for examining only a fraction of all possible subsets and stopping after the best subset is selected. The squared error criterion was suggested by Mellow (11) to be used in linear regression models.

The H-L algorithm was further investigated by

LeMotte and Hocking (12) for estimation of the linear regression coefficients to fit a surface of the form in (1.1). They have also introduced some other improvements to reduce the amount of computation. Later, Levasseur and Lewis (13) considered antenna design as a partial basis problem. They, too, applied a generalization of the H-L algorithm. The value of extending the work of Levasseur and Lewis to complex-valued functions was pointed out by D. W. Tufts. He also suggested applying the method to the synthesis of digital filters and to estimation of signal parameters in the presence of noise.

In the following chapter, we restate the methods of H-L and Levasseur and Lewis for complex subset selection to make it useful for the problems considered in this thesis. Digital filter synthesis will be discussed in chapter 3. And chapter 4 is devoted to estimation of the frequencies of sinusoidal signals by the method of complex subset selection.

## CHAPTER 2

### COMPLEX SUBSET SELECTION

#### 2.1- Introduction

Our objective is to extend an algorithm for partial basis selection to complex-valued basis functions. In this case, we have a large set of exponentials with imaginary arguments as basis functions. The desire is to approximate a given function by choosing a best subset of the basis set with minimum approximation error. The least squares error is used as the criterion for subset selection. The algorithm is designed such that usually only a fraction of all possible subsets is examined, and it comes to a halt as soon as a best subset is selected. Hence, considerable amounts of time and computation can usually be saved by not examining all possible subsets.

This algorithm is suitable for such problems as digital filter design and sinusoidal-tone estimation. This is due to the representation of the transfer function of a nonrecursive digital filter and the waveform of multiple sinusoidal signals by linear combinations of exponential functions with imaginary arguments.

Some work has been done on partial basis selection of real-valued functions by Levasseur and Lewis (13). Their

derivation is restated in the next section for complex partial basis selection.

### 2.2- The Algorithm

It is desired to approximate a given function  $D(x)$  by a best subset of  $M$  elements chosen from  $N$  linearly independent complex basis functions relative to some error norm. The least squares norm is discussed because of its later use, but other norms can be applied, as well.

Let us define a set of  $N$  complex basis functions as a vector  $H$ , and a subset with  $M$  ( $M < N$ ) elements selected from  $H$  as a vector  $U$ .

$$H^T = [h_1, h_2, \dots, h_N] \quad (2.2.1)$$

$$U^T = [u_1, u_2, \dots, u_M] \quad (2.2.2)$$

$$U \subseteq H$$

$$h_i(t) = \text{EXP}(J\omega_i t) \quad i=1, 2, \dots, N \quad (2.2.3a)$$

Or

$$h_i(v) = \text{EXP}(J\omega_i v) \quad i=1, 2, \dots, N \quad (2.2.3b)$$

The subset parameter vector is defined as

$$G = [g_1, g_2, \dots, g_M] \quad (2.2.4)$$

in which the components of G are distinct and for some value of i

$$g_n = \begin{cases} w_i & \text{for some value of } i \text{ when } h_i \\ t_i & \text{is defined by (2.2.3a)} \\ & \text{for some value of } i \text{ when } h_i \\ & \text{is define by (2.2.3b)} \end{cases}, \quad (2.2.5)$$

for  $n=1, 2, \dots, M$  and  $i=1, 2, \dots, N$

"T" means transpose, and  $J=\sqrt{-1}$ .

The basis functions can have two versions in (2.2.3) so that the approximation can be done either in the time domain or the frequency domain. One can replace time with position, and have basis functions related to location in space. The approximating function A(x) is of the form

$$A(x) = \sum_{k=1}^M c_k u_k = C^T U \quad (2.2.6)$$

in which the coefficient vector C is defined by

$$C^T = [c_1, c_2, \dots, c_M] \quad (2.2.7)$$

The error of approximation is defined to be

$$E(C, G) = \left\| D(x) - A(x) \right\|^2 = \int W(x) \left| D(x) - A(x) \right|^2 dx \quad (2.2.8)$$

and  $W(x)$  is a weight function which can be 1 or any suitable function. The error  $E(C, G)$  is to be minimized by choice of the subset parameter vector  $G$  and the coefficient vector  $C$ . Choosing  $G$  is equivalent to choosing a subset of  $M$  basis functions.

It is impractical to search over all  $\binom{N}{M}$  subsets to find the best one, especially when  $N$  is large which is the case in most situations. Hence, the following method is adopted which is based on the Hocking and Leslie (10) algorithm, as extended by Levasseur and Lewis (13) to a partial basis problem.

In the adopted method one searches for the potential basis functions which contribute the most to the error function by their deletion. That is, a set of error functions,  $E_i$ 's is obtained where  $E_i$  corresponds to the error due to a best approximation by using all basis functions excluding the  $i$ th one.

$$E_i = \min_{\{c\}} \left\| D(x) - \sum_{\substack{k=1 \\ k \neq i}}^N c_k h_k \right\| \quad (2.2.9)$$

Then, the basis functions are re-ordered in a way that

their corresponding  $E_i$ 's errors are in ascending order.

$$E_1 \leq E_2 \leq \dots \leq E_N \quad (2.2.10)$$

The basis functions corresponding to the large  $E_i$ 's are usually the potential basis functions which are good candidates to be kept at the begining of the search. Let  $r$  be a set of  $r = N - M$  basis functions to be deleted and let  $ER$  be the corresponding approximation error. Also, define  $p$  to be initially equal to  $r$ . Now, the search can be implemented in the following three steps.

STEP-1. Delete the  $r$  basis functions corresponding to  $E_i$ 's with subscripts less than, or equal to  $r$  ( $i \leq r$ ), and compute the minimum error  $ER$ . If  $ER \leq E_{P+1}$ , go to step-3. If  $ER > E_{P+1}$ , follow step-2.

STEP-2. Increment  $p$  by 1, and compute the error for all  $\binom{P}{r}$  combinations of deleting  $r$  out of  $p$  basis functions where deleted basis functions correspond to  $E_i$ 's with  $i \leq p$ . Let  $ER$  be the minimum error obtained from all possible  $\binom{P}{r}$  combinations. If  $ER \leq E_{P+1}$ , go to step-3; otherwise ( $ER > E_{P+1}$ ) re-execute step-2.

STEP-3. If  $ER \leq E_{P+1}$ , a best subset has been

achieved, and no further search is necessary. Hence the selected basis functions are those which have been kept for best approximation with minimum error either in step-1 or step-2.

NOTE: Step-1 is to start the search while step-2 is executed over and over until a best subset is obtained.

The search algorithm was outlined in the above three steps, but it was not explained why one stops whenever  $ER \leq E_{p+1}$ . Now, this will be discussed.

Assume  $p=r+q$  with  $1 \leq q < M$  to be a general case. Let  $I = \{1, 2, \dots, p\}$  be a set of  $p$  indices taken from the set  $\{1, 2, \dots, N\}$  with largest element  $p$ . Let  $I_q$  be a set of  $r$  indices selected from  $I$  which result in minimum error by their deletion, and let  $I' \subset \{1, 2, \dots, N\}$  be any set of  $r$  indices with an element  $i' > p+1$ . The intention is to show that if

$$ER = \min_{\{c\}} \left\| D - \sum_{\substack{i=1 \\ i \notin I_q}}^N c_i h_i \right\| \leq E_{p+1} \quad (2.2.11)$$

then,

$$ER \leq \left\| D - \sum_{\substack{i=1 \\ i \in I'}}^N c_i h_i \right\| = E_{I'} \quad (2.2.12)$$

for any  $I'$ .

The statement (2.2.12) can be proved as follows:

$$ER \leq E_{p+1}$$

$$E_{p+1} \leq E_{i'} \quad \text{BECAUSE } i' > p+1$$

$E_{i'}$  is the minimum error obtained by deleting  $i'$ 'th basis function plus some other ones, as well. Hence,

$$E_{i'} \leq E_{i'}$$

and,

$$ER \leq E_{p+1} \leq E_{i'} \leq E_{i'}$$

Thus, at any point, if the statement  $ER \leq E_{p+1}$  is satisfied, no further search would be necessary. However, if  $ER \leq E_{p+1}$  is never satisfied, all the possible subsets will be examined to select the best one.

To show the power of this algorithm, some numerical examples will be presented in the following chapters. Also, a windowing technique will be applied to improve the efficiency of the algorithm.

CHAPTER 3  
FILTER SYNTHESIS

### 3.1- Introduction

A finite-duration impulse response (FIR) filter can be represented with a set of coefficients and delays as depicted in Figure 3.1. The frequency response of such a filter is of the form shown by equation (3.1.1). The object is to design an FIR filter to have fewer coefficients than equivalent conventional FIR filters, while meeting the same performance objectives.

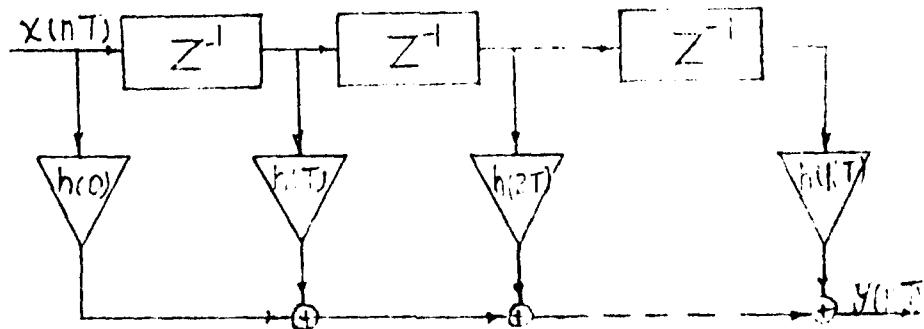


Figure 3.1

$$H(w) = \sum_{n=0}^N h(nT) \cdot \exp(j\omega nT) \quad (3.1.1)$$

Instead of processing all the input data over a given record length or a subset of uniformly spaced data

elements, we choose the best  $M$  data elements for achieving the desired performance. The fewer the coefficients, the fewer multiplications will be performed, since each coefficient corresponds to a multiplication directly. The complex subset selection method is used to achieve the goal of selecting the best subset of data elements, from a given interval of input data, to be processed by an FIR filter.

### 3.2- Formulation Of Design

The desired FIR filter is a lowpass filter(LPF) having a passband, a stopband, and a 'don't care' (3) region as depicted in figure 3.2. The desired frequency transfer function,  $D(\omega)$ , is real valued and symmetric about frequency zero. Assuming that sampled data is to be processed with a memory no greater than  $2qT$  seconds, or  $(2q+1)$  samples, we can approximate  $D(\omega)$  using all the available data by a FIR filter with transfer function  $A(\omega)$  given in formula (3.2.1).

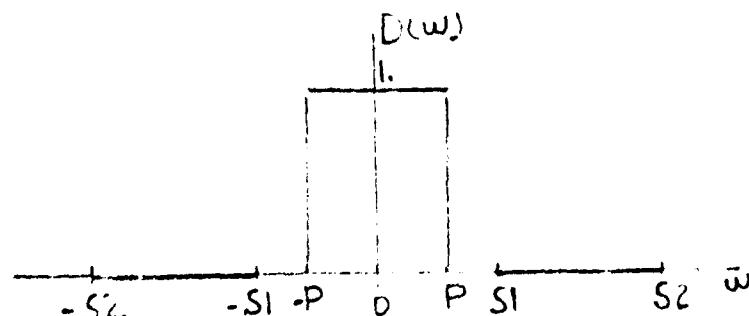


Figure 3.2

$$A(w) = \sum_{n=-q}^q a(nT) \exp(Jw nT) \quad (3.2.1)$$

There is no loss of generality by choosing zero phase for  $D(w)$ , because one can easily synthesize a linear phase filter by shifting the impulse response of this filter properly. Thus, the ideal filter is defined by equation (3.2.2).

$$D(w) = \begin{cases} 1 & , |w| \leq p \\ 0 & , s_1 < |w| < s_2 \\ \text{don't care, otherwise} & \end{cases} \quad (3.2.2)$$

Let the observation time be kept fixed. That is,  $q$  and  $T$  of (3.2.1) satisfy the formula

$$2qT = \text{constant.} \quad (3.2.3)$$

Using complex subset selection,  $D(w)$  can be approximated by a subset having fewer coefficients in the same observation time ( $M < 2q+1$ ) shown by equation (3.2.4),

$$H(w) = \sum_{n=1}^M h(n) \frac{\exp(Jw nT)}{n} \quad (3.2.4)$$

where the  $T_n$ 's may or may not be uniformly spaced. Using the method of complex subset selection, the large set of basis functions of the form  $\exp(Jw nT)$  are the potential

basis functions of (3.2.1) in which

$$\text{nth basis function} = \text{EXP}(J\omega_n T) \quad (3.2.5)$$

$$n = -q, \dots, -1, 0, 1, \dots, q$$

$$N = 2q + 1. \quad (3.2.6)$$

A best subset having M coefficients is selected from these N basis functions such that the least squares error between  $D(w)$  and  $H(w)$  is minimized, i.e.,

$$\text{Min. Error} = \int W(w) \left| D(w) - \sum_{n=1}^M h(n) \frac{\text{EXP}(J\omega_n T)}{n} \right|^2 dw \\ \{h(n), T_n\} \quad (3.2.7)$$

where

$$W(w) = \begin{cases} 1 & , w | \leq p \text{ and } s_1 \leq w \leq s_2 \\ 0 & , p < w < s_1 \end{cases} \quad (3.2.8)$$

and the  $T_n$ 's are selected out of the set  $(-qT, \dots, -T, 0, T, \dots, qT)$ .

From the orthogonality principle (14) for optimum choice of the  $h(n)$ 's, for given values of the  $T_n$ 's, it follows that

$$\int W(w) (D(w) - \sum_{n=1}^M h(n) \frac{\text{EXP}(J\omega_n T)}{n}) \cdot \text{EXP}(-J\omega_k T) dw = 0.$$

$$\text{for } k = 1, 2, \dots, M \quad (3.2.9)$$

which can be rewritten as a set of M linear equations of the form

$$AH = B \quad (3.2.10)$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1M} \\ a_{21} & \dots & & \\ a_{M1} & \dots & \dots & a_{MM} \end{bmatrix} \quad (3.2.11)$$

$$H^T = [h(1), h(2), \dots, h(M)] \quad (3.2.12)$$

$$B^T = [b_1, b_2, \dots, b_M] \quad (3.2.13)$$

where

$$a_{ij} = \begin{cases} \frac{(2/(T_i - T_j))(\sin(s_2(T_i - T_j)) - \sin(s_1(T_i - T_j))}{i-j} + \sin(p(T_i - T_j)) & i \neq j \\ 2(s_2 - s_1 + p) & i=j \end{cases} \quad (3.2.14)$$

$$b_i = \begin{cases} (2/(T_i)) \cdot \sin(pT_i) & T_i \neq 0 \\ 2p & T_i = 0 \end{cases} \quad (3.2.15)$$

$$H = A^{-1} B \quad (3.2.16)$$

For the detailed derivation of the equations (3.2.9) through (3.2.17), see appendix A.

Since  $A$  and  $B$  are both real valued matrices, the solution to the equation (3.2.16) results in real coefficients for the filter (i.e.  $h(n)$ 's are all real). Solving for  $H$  and substituting in the error function of (3.2.7) is equivalent to simplifying (3.2.7) using (3.2.9). The result is

$$\text{Min. Error} = 2p - 2 \sum_{n=1}^M h(n) \frac{\sin(pT_n)}{n} / T_n \quad (3.2.17)$$

in which  $\{h(n)\}$  satisfies (3.2.9), or equivalently (3.2.16).

The results obtained here will be compared with a filter having uniformly spaced values of  $T$  with frequency response given by (3.3.1), and we claim that the complex subset selection will always result in a better approximation to a desired ideal filter, or at least an approximation as good as the one obtained by the uniformly-spaced method. The uniformly-spaced method is a search for only a constrained set of possible solutions, and many other possibilities are simply ignored. On the other hand, whenever complex subset selection is used, all possible approximations are considered. If it happens that

the best solution is uniformly spaced, it will be included as one of the possible solutions to be examined.

### 3.3- Filter With Uniformly-spaced Time

#### Samples

The filter obtained by the complex subset selection will be compared with another filter which uses uniformly-spaced time samples. For this filter, the observation time ( $2qT$ ) is fixed and the number of the coefficients ( $M$ ) are the same as before. But, the basis function parameters must be spaced uniformly from an initial location ' $c$ ' as illustrated in figure 3.3.

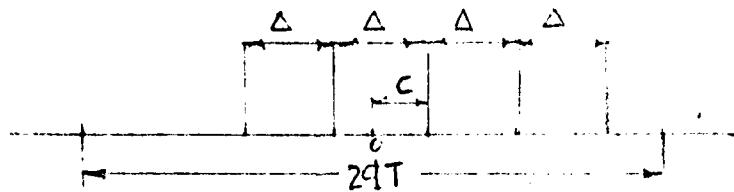


Figure 3.3

The frequency response of this filter is of the form shown by equation (3.3.1), assuming that  $M$  is an odd integer of the form  $M=2l+1$

$$H_u(w) = \sum_{n=-l}^{l} h(n) \exp(j(n\Delta+c)w) \quad (3.3.1)$$

If, for fixed values of  $\Delta$  and  $c$ , we minimize the error given by

$$\text{Error} = \int W(w) |D(w) - H_U(w)|^2 dw \quad (3.3.2)$$

by choice of the coefficients  $h(n)$ , we obtain, using the orthogonality principle (14), the set of simultaneous linear equations

$$A' H = B' \quad (3.3.3)$$

in which  $A'$  and  $B'$  can be evaluated from (3.2.9) by letting  $T_n = n\Delta + c$ . The coefficients  $\{h_U(n)\}$  are also real since  $A'$  and  $B'$  are real-valued matrices.

In the search for the best  $c$ , it was found numerically that the best  $c$  is zero for the best  $\Delta$  for the cases tried (four examples are illustrated in table 3.1 where the  $C$  was searched in the interval 0.0 to 3.0 with increment .1, and best  $\Delta$  was searched in the interval 0. to 2.5 with increment .1). Therefore, we tried to prove this analytically. However, after investigating many different methods, we could only show that  $C=0$  is a good candidate for best  $C$ , and could not prove its uniqueness. One method is to differentiate the error function with respect to  $c$ , and set it to zero. The error function is of the form

$$\begin{aligned}
 E &= 2P - 2 \sum_{n=-1}^1 h(n) \cdot \sin((n\Delta+c)P) / (n\Delta+c) \\
 &= 2P - B' A'^{-1} B' \\
 \end{aligned} \tag{3.3.4}$$

in which  $A'$  and  $B'$  are matrices from equation (3.3.3). The  $B'$  matrix is dependent on  $c$ , whereas the  $A'$  matrix is independent of  $c$ . Differentiating (3.3.4) with respect to  $c$  and setting it to zero, we have

$$\begin{aligned}
 \frac{dE}{dc} &= - (dB'^T / dc) A'^{-1} B' - B'^T A'^{-1} (dB'/dc) \\
 &= - 2B'^T A'^{-1} (dB'/dc) = 0 \\
 \end{aligned} \tag{3.3.5}$$

$$\begin{aligned}
 b_n &= 2 \sin((n\Delta+c)P) / (n\Delta+c) \\
 \frac{db}{dc} &= 2 ( P \cos((n\Delta+c)P) / (n\Delta+c) \\
 &\quad - \sin((n\Delta+c)P) / ((n\Delta+c)^2) ) \\
 \end{aligned} \tag{3.3.6}$$

$$\begin{aligned}
 \frac{dE}{dc} &= - 8 \sum_{n=-1}^1 \sum_{m=-1}^1 \sin((n\Delta+c)P) / (n\Delta+c) \\
 &\quad [ P \cos((m\Delta+c)P) / (m\Delta+c) - \sin((m\Delta+c)P) / ((m\Delta+c)^2) ] \\
 &\quad \cdot a'(n, m) \\
 \end{aligned} \tag{3.3.7}$$

where  $a'(n, m)$  are elements of  $A'^{-1}$ . Since  $A'$  is a toeplitz

matrix (see equations (3.2.11) and (3.2.14)), it follows that

$$a'(n, -m) = a'(-n, m) \quad (3.3.8)$$

and

$$\begin{aligned} dE/dc &= \sum_{n=1}^1 \sum_{m=-1}^1 \sin(n\Delta P)/n\Delta \cdot (P \cos(-m\Delta P)/(-m\Delta) \\ &\quad - \sin(-m\Delta P)/((-m\Delta)^2)) \cdot a'(n, -m) \\ &\quad + \sin(-n\Delta P)/n\Delta \cdot (P \cos(m\Delta P)/m\Delta) \\ &\quad - \sin(m\Delta P)/((m\Delta)^2) \cdot a'(-n, m) = 0. \quad (3.3.9) \end{aligned}$$

Thus  $c=0$  is a solution of (3.3.5), because all the terms in (3.3.7) cancel one another for  $c=0$ . Since we have not been able to show analytically the uniqueness of  $c$ , we search for the best  $c$  as we do for the best  $\Delta$ .

In the next section, some examples are presented in order to explain the differences of the two filters.

#### 3.4- Results And Examples

Presented here are the results of the comparison of the two filters discussed in sections 3.2 and 3.3. Many examples have been investigated, and only two will be presented here to show the differences in the two filters. In both examples, 21 basis functions are used with arguments  $w_{nT}$ , ( $n=-q, \dots, 0, \dots, q$ , and  $T=.5$ ), and the

subset size is 5 ( $M=5$ ). The results are tabulated in tables 3.2 and 3.3 for which the following observations are made. Of course, these observations are drawn from the investigation of a large set of examples.

- a) The least squares error is smaller for the subset selection filter than the uniformly spaced parameter filter.
- b) The magnitude response is flatter in the passband region in the subset selection filter case.
- c) The magnitude response rolls off sharply in the transition-band for the subset selection filter.
- d) However, the maximum sidelobe level is smaller for the uniform parameter filter.

The above observations can be seen from tables 3.2 and 3.3 and also from magnitude response plots of figure 3.4 and 3.5. The frequency response is symmetric about frequency  $1/2T$ , where  $T$  is the sampling period. Since the smallest sampling period for 21 basis functions is  $T=.5$ , all frequency responses are plotted over frequencies 0 through 1 hertz ( $1/2(.5) = 1$ ) .

It was also noticed that the nonzero coefficient locations are moved toward (or away from) the midpoint by

increasing (or decreasing) the passband width. That is, the coefficient locations are around the midpoint for a wide passband, and farther away for a narrow passband. The plot in figure 3.6 shows this effect where the abscissa is the second term in equation (3.3.2) ( $\sum_{n=0}^N h_u(n) \cdot \sin(n\Delta) p/n\delta$ ), and the ordinate is  $\Delta$ . Each curve is for a different passband value while each peak represents the best  $\Delta$  for that specific passband value.

TABLE 3.1a

M=5  
 $P=.15$     $S1=.2$     $S2=.5$

C	ERRCR FOR	ERROR FOR	ERROR FOR
	$\Delta=0.5$	$\Delta=1.0$	$\Delta=1.5$
0.0	.144201	.092526	.162134
0.1	.147396	.093267	.161562
0.2	.156949	.095505	.159873
0.3	.172754	.099286	.157136
0.4	.194671	.104690	.153471
0.5	.222486	.111818	.149040
0.6	.255916	.120801	.144044
0.7	.294664	.131788	.138728
0.8	.338348	.144936	.133360
0.9	.386559	.160419	.128241
1.0	.438853	.178403	.123694
1.1	.494751	.199062	.120051
1.2	.553746	.222544	.117660
1.3	.61326	.248993	.116866
1.4	.678959	.278519	.118011
1.5	.744113	.311209	.121422
1.6	.810252	.347113	.127416
1.7	.876849	.386238	.136278
1.8	.943399	.428554	.148267
1.9	1.00941	.473985	.163603
2.0	1.07444	.522403	.182472
Best C	0.0	0.0	1.3
ERROR	.144201	.092526	.116866

TABLE 3.1b

**M=5**  
**P=.1 S1=.2 S2=.5**

C	ERROR FOR $\Delta=0.5$	ERROR FOR $\Delta=1.0$	ERROR FOR $\Delta=1.5$
0.0	.103650	.032061	.036306
0.1	.104927	.032794	.036345
0.2	.108742	.034986	.036466
0.3	.115084	.038536	.036681
0.4	.123900	.043731	.037008
0.5	.135168	.050260	.037473
0.6	.148798	.058206	.038109
0.7	.164734	.067545	.038957
0.8	.182873	.078256	.040061
0.9	.203121	.090311	.041470
1.0	.225369	.103673	.043241
1.1	.249488	.118313	.045435
1.2	.275352	.134188	.048111
1.3	.302826	.151259	.051339
1.4	.331759	.169477	.055175
1.5	.361995	.188798	.059697
1.6	.393391	.209166	.064967
1.7	.425772	.230528	.071049
1.8	.458985	.252826	.078011
1.9	.492862	.276001	.085911
2.0	.527239	.299990	.094807
Best C	0.0	0.0	0.0
Error	.103650	.032061	.036306

TABLE 3.1C

C	ERROR FOR	ERROR FOR	ERROR FOR
	$\Delta = 0.5$	$\Delta = 1.0$	$\Delta = 1.5$
0.0	.093154	.068523	.154084
0.1	.096298	.068842	.153555
0.2	.105711	.069827	.152005
0.3	.121291	.071544	.149489
0.4	.146899	.074114	.156113
0.5	.170340	.077695	.142028
0.6	.203352	.082489	.137417
0.7	.241646	.088724	.132498
0.8	.284865	.096650	.127518
0.9	.332622	.106542	.122751
1.0	.384496	.118675	.118493
1.1	.440033	.133334	.115049
1.2	.498750	.150784	.112741
1.3	.560160	.171278	.111889
1.4	.623744	.195045	.112813
1.5	.688995	.222276	.115825
1.6	.755392	.253120	.129281
1.7	.822430	.287677	.129281
1.8	.889607	.326001	.140256
1.9	.956452	.368084	.154369
2.0	1.02250	.413856	.171806
Best C	0.0	0.0	1.3
Error	.093154	.068523	.111889

TABLE 3.1d

$M=5$   
 $P=.1$     $S_1=.225$     $S_2=.5$

C	ERROR FOR $\Delta=0.5$	ERROR FOR $\Delta=1.0$	ERROR FOR $\Delta=1.5$
0.0	.062489	.018249	.036055
0.1	.063793	.018802	.036085
0.2	.067678	.020460	.036176
0.3	.074139	.023223	.036341
0.4	.083124	.027090	.036599
0.5	.094604	.032058	.036975
0.6	.108495	.038125	.037506
0.7	.124734	.045289	.038225
0.8	.143224	.053546	.039182
0.9	.263868	.062889	.040426
1.0	.186553	.073311	.042016
1.1	.211153	.084809	.044006
1.2	.237535	.097367	.046464
1.3	.265566	.110980	.049453
1.4	.395094	.125629	.053040
1.5	.325968	.141302	.057292
1.6	.358029	.157980	.062277
1.7	.391109	.175643	.068061
1.8	.425056	.194265	.074711
1.9	.459692	.213823	.082287
2.0	.494855	.234284	.090847
Best C	0.0	0.0	0.0
Error	.062489	.018249	.036055

TABLE 3.2

	$N=21$ $f_P = .15$	$M=5$ $f_{S1} = .2$	$f_{S2} = .5$	
	Filter with N coefficients	Uniform parameter filter	Subset selection filter	
coefficient locations	$n_T$ $n=-10, \dots, 10$	$n_A$ $0, \pm 1.0, \pm 2.0$	$T_n$	$0, \pm 1.5, \pm 4.5$
Max. passband level (db)	.677	1.3	.73	
magnitude (db) at $f_P$	-2.6	-4.5	-3.5	
magnitude (db) at $f_{S1}$	-12.9	-10.1	-17.1	
Max. sidelobe level (db)	-25.2	-24.3	-21.0	
least squares error	.2098836E-01	.92527E-01	.7460785E-01	

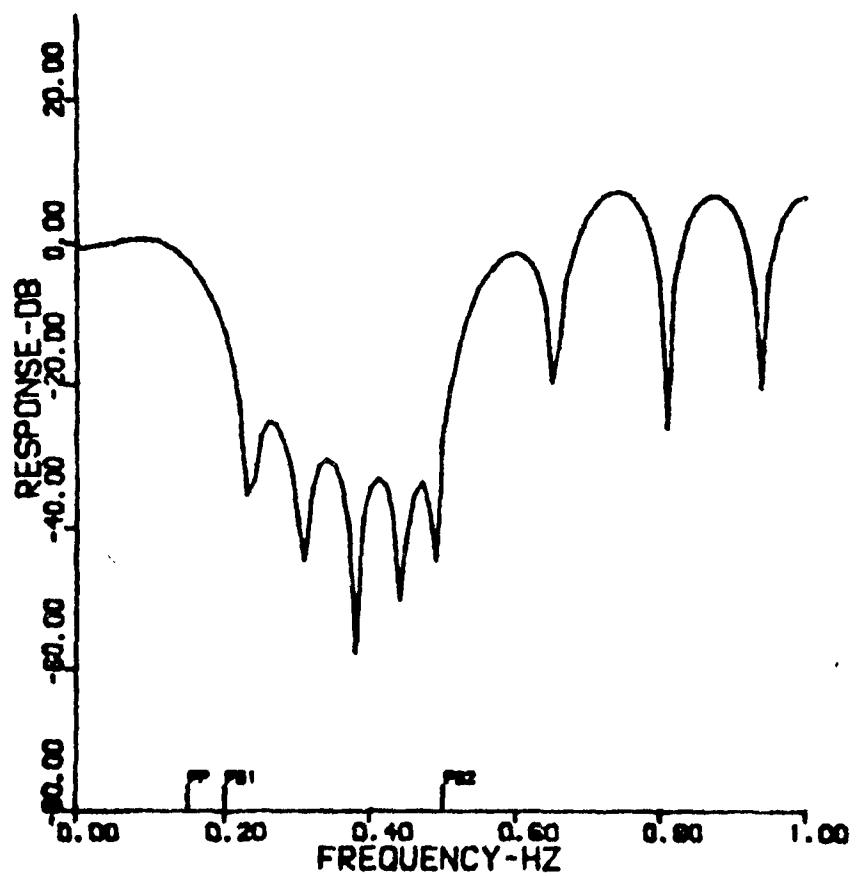


Figure 3.4a- Frequency response of filter  
having 21 coefficients with  $f_p = .15$ ,  $f_{s1} = .2$ ,  
and  $f_{s2} = .5$ .

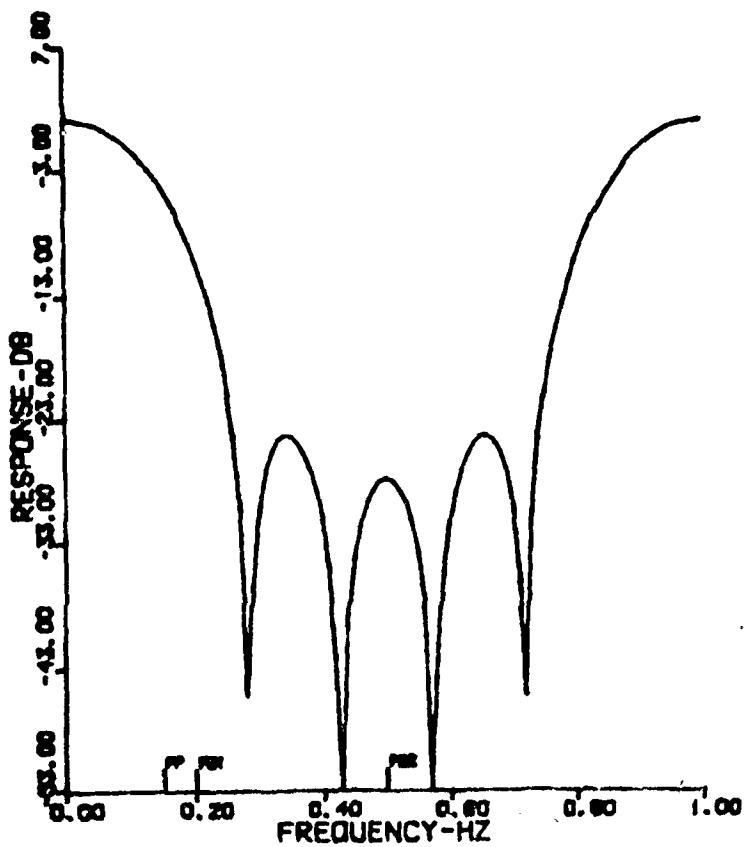


Figure 3.4b- Frequency response of uniform parameter filter having 5 coefficients with  $f_p = .15$ ,  $f_{s1} = .2$ , and  $f_{s2} = .5$ .

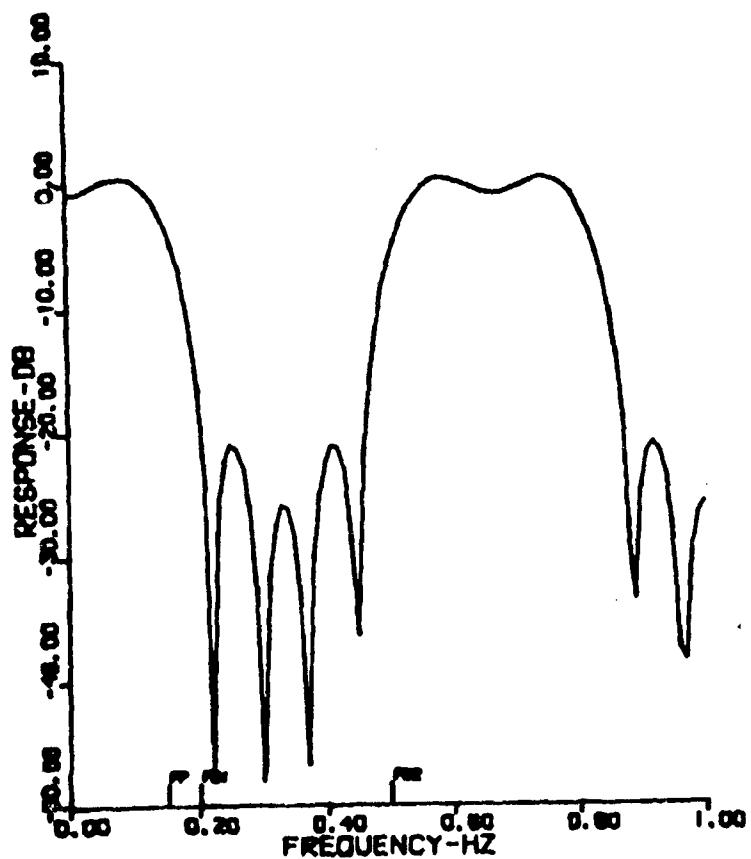


Figure 3.4c- Frequency response of subset selection filter having 5 coefficients with  $f_p = .15$ ,  $f_{s1} = .2$ , and  $f_{s2} = .5$ .

TABLE 3.3

	$N=21$ $f_P = .20$	$M=5$ $f_{S1} = .3$ $f_{S2} = .5$	
	Filter with N coefficients	Uniform parameter filter	Subset selection filter
coefficient location	$n_T$ $n=-10, \dots, 10$	$n_A$ $0, \pm 0.5, \pm 1.0$	$T_B$ $0, \pm 1.0, \pm 3.0$
Max. passband level (db)	.30	1.2	.47
magnitude (db) at $f_P$	-.93	-3.8	-1.7
magnitude (db) at $f_{S1}$	-21.8	-12.4	-15.1
Max. sidelobe level (db)	-32.7	-27.0	-25.1
least squares error	.2613000E-02	.79270E-01	.1487637E-01

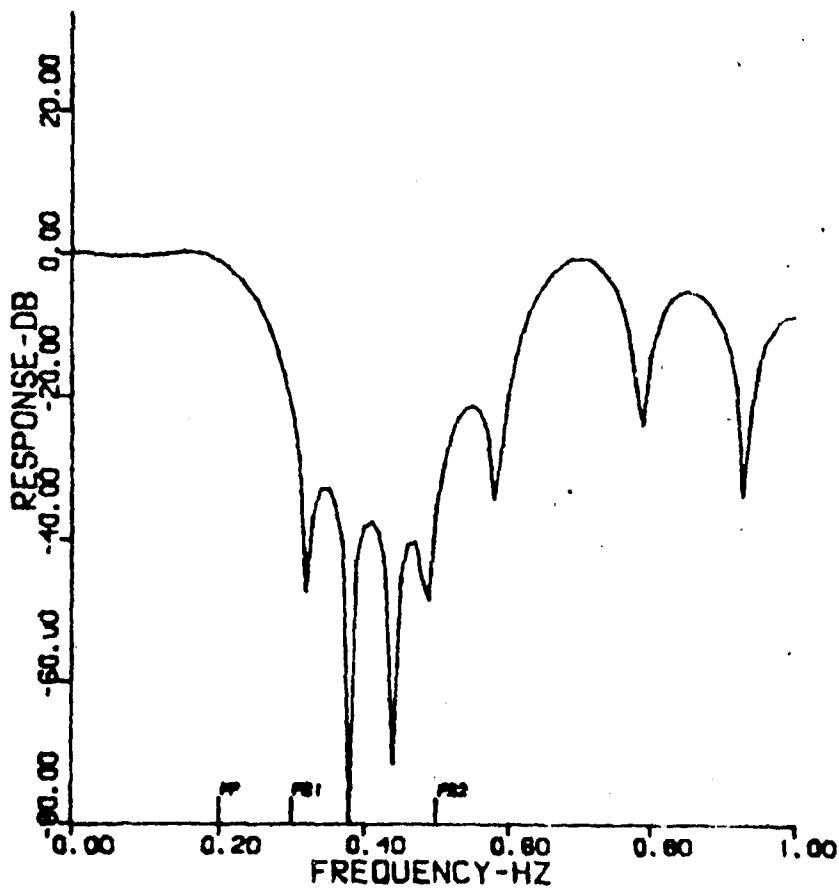


Figure 3.5a- Frequency response of filter having 21 coefficients with  $f_p = .2$ ,  $f_{s1} = .3$ , and  $f_{s2} = .5$ .

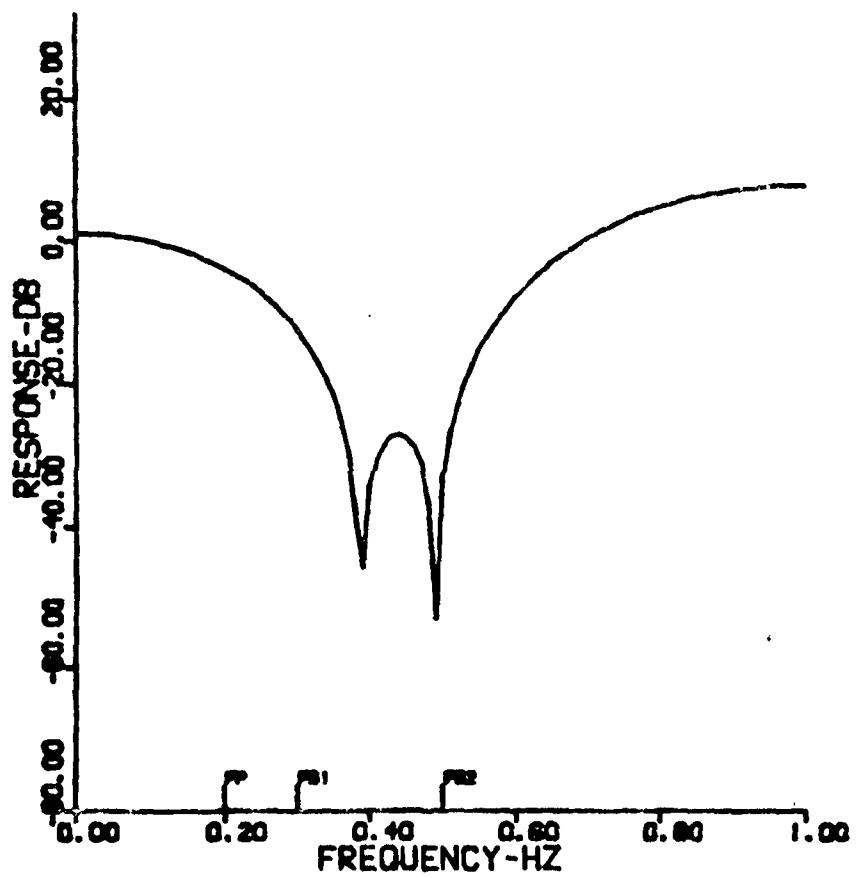


Figure 3.5b- Frequency response of uniform parameter filter having 5 coefficients with  $f_p = .2$ ,  $f_{s1} = .3$ , and  $f_{s2} = .5$ .

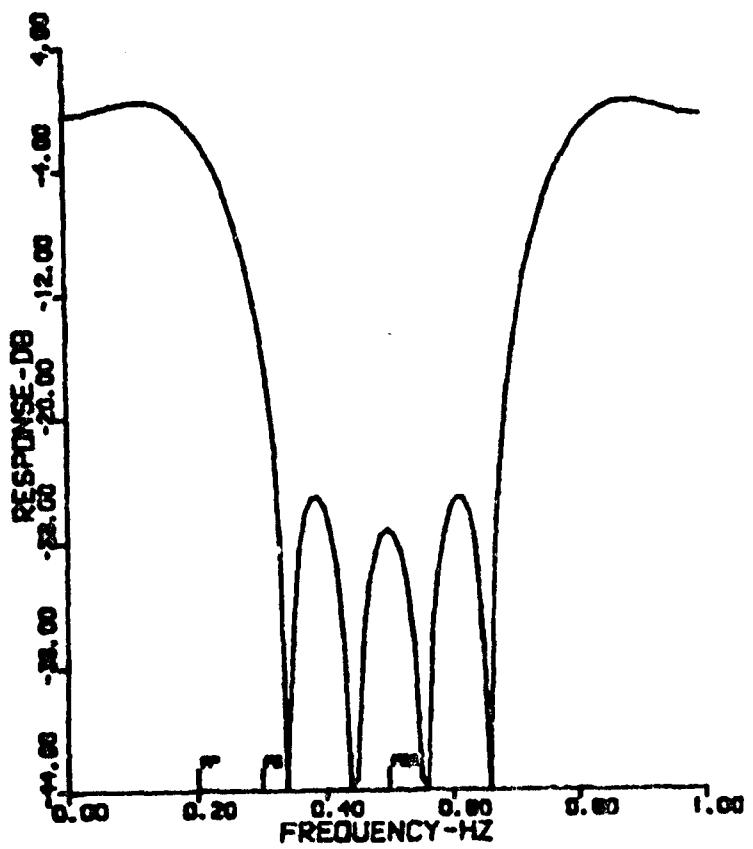


Figure 3.5c- Frequency response of subset selection filter having 5 coefficients with  $f_p = .2$ ,  $f_{s1} = .3$ , and  $f_{s2} = .5$ .

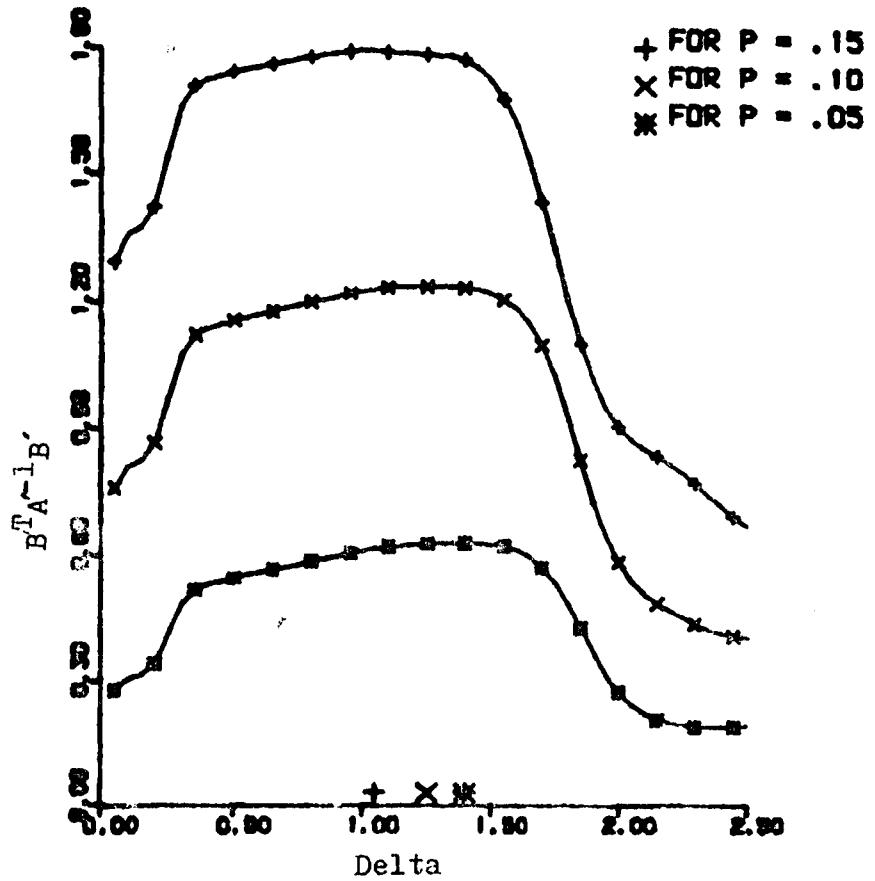


Figure 3.6-  $B'^T A'^{-1} B'$  (variable part of the error function) for different values of delta and passband width.

CHAPTER 4  
ESTIMATION OF THE FREQUENCIES  
OF SINUSOIDS

4.1- Introduction

This chapter is devoted to the problem of estimation of the frequencies of sinusoidal tones by the complex subset selection method. In this case, the function to be approximated contains some tones with specific amplitudes and frequencies in the presence or absence of noise. We consider the case of two sinusoidal tones, but the method can be extended to more tones at the cost of more computation.

The framework of this problem will be developed in section 4.2; while the frequency estimation will be presented in section 4.3. The presence of Gaussian noise in the data will be investigated in section 4.4, and finally section 4.5 will contain some numerical results.

4.2- Two Tone Approximation

The signal to be approximated contains two sinusoidal tones with each tone having amplitude and frequency  $(A_1, F_1)$  and  $(A_2, F_2)$ , respectively.  $A_1$  and  $A_2$  can

be either real or complex.

$$S(t) = A_1 \exp(J2\pi f_1 t) + A_2 \exp(J2\pi f_2 t) \quad (4.2.1)$$

A data vector  $y$  is formed from  $N$  samples of above function (4.2.2).

$$y^T = [y(0), y(1), \dots, y(N-1)] \quad (4.2.2)$$

$$y(n) = S(nT)$$

The complex exponential basis functions used here are

$h_I(t), h_2(t), \dots, h_L(t)$  with arguments of the form  $2\pi f_I t$ , i.e.,

$$h_I(t) = \exp(J2\pi f_I t) \quad \text{for } I=1, 2, \dots, L \quad (4.2.3)$$

A vector is formed for each basis function with  $N$  samples taken from each of them.

$$h_I^T = [h_I(0), h_I(1), \dots, h_I(N-1)] \quad \text{for } I=1, 2, \dots, L$$

with

$$(4.2.4)$$

$$h_I(n) = \exp(J2\pi f_I n)$$

and

$$0 < f_1 < f_2 < \dots < f_L < 1$$

Due to the periodicity of exponentials with an imaginary arguments,  $f_L$  should be less than one. That is

$$\text{EXP}(J2\pi f_L n) = \text{EXP}(J2\pi(f_L - 1)n)$$

Having  $H_I$ 's for basis functions, we want to approximate the data vector  $Y$  by the complex subset selection method with minimum least squares error. In another words, we want to choose  $M$  functions from the  $L$  basis functions to best represent the data vector,  $Y$  in order to minimize the sum of the squares of the errors.

Assume  $Y$  is the data approximation vector of the form

$$Y = \sum_{k=1}^M c_k u_k = UC \quad (4.2.5)$$

where  $U$  is a vector formed from  $M$ -column vectors  $U_k$ 's and  $C$  is a  $M$ -dimensional complex coefficient vector.

$$C^T = [c_1, c_2, \dots, c_M] \quad (4.2.6)$$

$$U = [U_1; U_2; \dots; U_M] \quad (4.2.7)$$

$$U_k^T = [u_k(0), u_k(1), \dots, u_k(N-1)] \quad (4.2.8)$$

$$u_k(n) = \text{EXP}(J\omega_k n) \quad (4.2.9)$$

$$\mathbf{w} = \begin{bmatrix} w_1, w_2, \dots, w_M \end{bmatrix}$$

Note  $U_K$ 's are chosen from  $H_I$ 's, and  $w_K$ 's are selected from  $2\pi f_I$ 's. The error function is

$$\text{Min. Error } E = \|Y - \hat{Y}\|_{\{\mathbf{C}, \mathbf{w}\}} \quad (4.2.10)$$

where  $\|\cdot\|$  means the L2 norm of the vector.

To minimize the error one has to find the optimum coefficient vector and frequency vector to be used in specifying  $\hat{Y}$ .

For the optimum coefficient vector, the orthogonality principle indicate; that  $Y - \hat{Y}$  must be orthogonal to every  $U_l^*$  with  $l=1, \dots, M$ . (\* means conjugate transpose).

$$U_l^* (Y - \hat{Y}) = 0 \quad \text{for } l=1, 2, \dots, M \quad (4.2.11)$$

$$U_l^* \hat{Y} = U_l^* Y \quad \text{for } l=1, 2, \dots, M \quad (4.2.12)$$

$$U_l^* U_l C = U_l^* Y \quad \text{for } l=1, 2, \dots, M \quad (4.2.13)$$

Let  $A = U^* U$ , and  $B = U^* Y$ , THEN

$$AC = B \quad (4.2.14)$$

where

$$A = \begin{bmatrix} * & * & * \\ U_{11} & U_{12} & \dots & U_{1M} \\ 1 & 1 & 2 & \dots & M \\ \dots & & & & \\ * & & & * \\ U_{M1} & \dots & \dots & U_{MM} \end{bmatrix}$$

with

$$U_{1k}^* = \sum_{n=0}^{N-1} \exp(J(w_k - w_1)n)$$

$$B^T = \begin{bmatrix} * & * & * \\ U_1 Y, U_2 Y, \dots, U_M Y \\ 1 & 2 & M \end{bmatrix} \quad (4.2.15)$$

$$b_1^* = U_1 Y = \sum_{n=0}^{N-1} y(n) \exp(-Jw_1 n) \quad (4.2.16)$$

The solution of the linear equation (4.2.14) will give the optimum coefficient vector C. Thus, the error function becomes

$$\begin{aligned} E^2 &= \|Y - \hat{Y}\|^2 = (Y - \hat{Y})^* (Y - \hat{Y}) \\ &= Y^* (Y - \hat{Y}) - \hat{Y}^* (Y - \hat{Y}) \\ &= Y^* (Y - \hat{Y}) - C^* U^* (Y - \hat{Y}) \end{aligned} \quad (4.2.17)$$

The right side of the above equation can be simplified by

using equation (4.2.11).

$$\begin{aligned}
 E^2 &= Y^* (Y - \hat{Y}) = |Y|^2 - Y^* \hat{Y} \\
 &= |Y|^2 - Y^* UC = |Y|^2 - B^* C \\
 &= |Y|^2 - B A^{-1} B
 \end{aligned} \tag{4.2.18}$$

$$E^2 = |Y|^2 - B^* C = \sum_{n=0}^{N-1} |y(n)|^2 - \sum_{k=1}^M \frac{c_k}{k} y(n) \exp(-j\omega_n k) \tag{4.2.19}$$

Let us choose the frequencies  $f_i$ 's of the basis functions to be uniformly spaced. Hence, the functions will be of the form

$$h_I(n) = \exp(j2\pi f_I n) \tag{4.2.20}$$

such that  $\Delta f = 1/L$  is the difference between two consecutive grid points. The goodness of the approximation is dependent on  $L$  and  $N$ , namely, the number of the basis functions and the number of the samples. Since  $\Delta f = 1/L$ , any increase in  $L$  will result in having denser grid points which in turn will increase the possibility that  $f_1$  and  $f_2$  to be on the grid points.  $f_1$  and  $f_2$  being on the grid points means that two of the basis functions can exactly approximate the data vector. On the other hand, large  $N$  means having more data points to process which will also result in better

approximation. But L cannot get too large, because the amount of the computation will get impractical for large L. Thus , the windowing technique described below will be applied whenever very dense grid points are needed.

#### 4.3- Frequency Estimation

In some applications one has to estimate the frequencies of the tones as accurately as possible. In those cases, we set the number of the basis functions M in the selected subset to be equal the number of the frequencies contained in the signal. For example, when we are estimating two sinusoidal tones we set M=2. But a very dense set of grid points is needed in order for the estimation to be accurate, and having a very dense grid points is impractical computationally since the amount of the computation will increase significantly as the number of the grid points becomes large. To get around this deficiency, we introduce the windowing technique below.

The windowing technique is applied whenever the grid points (number of basis functions) are large, and the search for a best subset needs too much computer processing. In those cases, a best subset is selected from a less dense grid as a first step. In second step, a denser grid is chosen around the selected subset grid points of step one. As step three, a new best subset is selected from the new grid points of step two. Then, step two and three

repeated until no changes are noticed in the selected subset and the error function. One should notice that in the first attempt, potential basis functions are selected where in the second attempt a new set of basis functions are established which are densely populated around those selected potential basis functions, and the best subset is selected. Graphically, the windows in Figure 4.1 indicate where the new densely populated grid points are to be set.

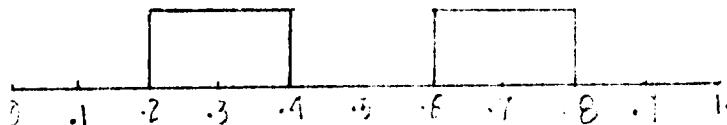


Figure 4.1

To clarify the windowing technique, let us take a look at the two-tone signal, with frequencies  $f_1=.25$  and  $f_2=.725$ . Choose only ten grid points uniformly spaced from zero to one, for the first step. Thus, the basis functions are

$$\sum_I h_I(n) = \text{EXP}(j2\pi f_I n) \quad I=0, 1, \dots, 9$$

with  $L=10$ , and grid points

$$(0., 0.1, 0.2, \dots, 0.9)$$

of which 2 are selected as potential functions. Assume

$p_{th} = .3$  and  $q_{th} = .7$  are selected. In the next step, we use the following ten new basis functions around the  $p_{th}$ , and  $q_{th}$ :

$$h_I(n) = \text{EXP}(j2\pi(p + \frac{\Delta f}{2}(I-2))n) \quad I=0,1,2,3,4$$

$$h_I(n) = \text{EXP}(j2\pi(q + \frac{\Delta f}{2}(I-7))n) \quad I=5,6,7,8,9$$

with  $L=10$ , and grid points

$$(0.2, 0.25, 0.3, 0.35, 0.4, 0.6, 0.65, 0.7, 0.75, 0.8).$$

Note the new grid points are one-half of the previous step. The complex subset selection algorithm is again used to select two potential functions from these new basis functions. In this case, second and ninth ( $p=.25, q=.75$ ) are selected. Again, choose another ten basis functions with grid points around .25, and .75

$$h_I(n) = \text{EXP}(j2\pi(p + \frac{\Delta f}{4}(I-2))n) \quad I=0,1,\dots,4$$

$$h_I(n) = \text{EXP}(j2\pi(q + \frac{\Delta f}{4}(I-7))n) \quad I=5,6,\dots,9$$

with  $L=10$ , and new grid points

$$(0.2, 0.225, 0.25, 0.275, 0.3, 0.675, 0.7, 0.725, 0.75, 0.775).$$

Continue as before until the selected frequencies and the error do not change very much for a denser grid points. In this case .25 and .725 are the selected frequencies . The above example is listed in Table 4.4.

If at any time, two windows overlap, that is, two selected frequencies are close to one another, use one wide window instead of two as depicted below.

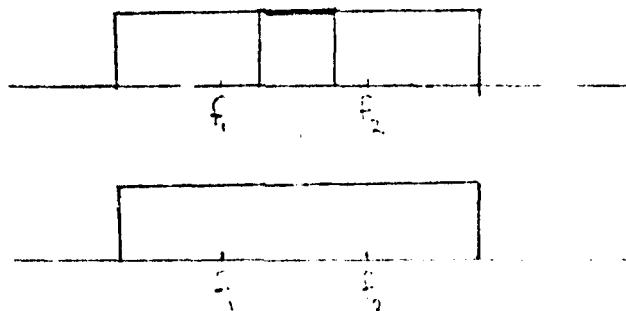


Figure 4.2

The program listings for the two frequency estimation by subset selection and windowing technique are in appendix B.

#### 4.4- The Effect Of Additive Noise

The approximation of a two-tone signal was investigated in section 4.2. But, it was for deterministic signals, and in almost all practical applications the received signal is perturbed by some kind of noise. In here, we investigate the additive Gaussian noise which is the general case for most systems.

Recall the data vector  $\mathbf{Y}$  (4.2.2) which was obtained from a deterministic signal, only. Now we form a new data vector from signal vector  $\mathbf{S}$  with noise vector  $\mathbf{V}$  added to it.

$$\mathbf{Y} = \mathbf{S} + \mathbf{V} \quad (4.4.1)$$

$$\mathbf{v}^T = [v(0), v(1), \dots, v(V-1)] \quad (4.4.2)$$

Note that each vector in (4.4.1) is  $N$ -dimensional. Assume  $\mathbf{V}$  consists of complex, independent, zero-mean Gaussian noise components with known variance.

By a close look at linear equation (4.2.1), one will realize that the noise will affect the  $\mathbf{B}$  vector, only. That is, each row of  $\mathbf{B}$  is changed by a amount due to the inner product of the noise vector and the corresponding  $\mathbf{U}$  vector (where subscript corresponds to  $I$ the row.)

$$\mathbf{B} = \begin{matrix} * & * & * \\ \mathbf{U} \mathbf{Y} & = & \mathbf{U} \mathbf{S} + \mathbf{U} \mathbf{V} \end{matrix} \quad (4.4.3)$$

Therefore, the same procedure as before is used to find the coefficient vector, selected basis function parameters, and the error function with new data vector.

The effects of Gaussian noise in tone parameter estimation has been well investigated by David C. Rife (8). He has used a multiparameter generalization of the Cramer-Rao bound (C-R bound) to establish lower bounds for variances. He has also shown that the single-tone bounds

are the minimum values attained by two-tone bounds, and they are attained when the two tones have wide frequency separation. Thus, one can use the single-tone bounds (4.4.4) for two-tone bounds whenever two frequencies are far apart.

$$\text{var}(\hat{w}_i) \geq (12\hat{\sigma}_i^2)^2 / (b_i^2 T N(N-1)) \quad (4.4.4)$$

where

$\hat{w}_i$  frequency estimate in radians

$\hat{\sigma}_i^2$  noise variance

$b_i$  tone amplitude

$T$  time between two samples

$N$  number of samples

In the next section, this C-R bound is compared with the measured accuracy in frequency estimation which is obtained by applying the complex subset selection algorithm to simulated data.

#### 4.5 Results and Examples

Presented here are numerical results for tone estimation by the complex subset selection method. The examples are chosen from three groups to show the power of

the complex subset selection method, the windowing technique for frequency estimation, and the effect of the noise.

In the first example, the signal contains two tones with amplitudes and frequencies ( $A_1=1.$ ,  $A_2=1.$ ), ( $F_1=.2$ ,  $F_2=.4$ ), respectively. Twenty basis functions were used where the frequency separation of the basis functions was .05 (  $f=.05$  , grid point separation ). Subsets of size 5 ( $M=5$ ) were searched, and the best one was selected after only examining 136 subsets. This is a small fraction of all possible subsets  $(20!/(15!5!) = 15504)$ . However, the two frequencies were exactly on the grid points. This example is illustrated in table 4.1 along with the re-ordered  $E_i$  errors. For the second example, the amplitudes and frequencies were ( $A_1=1.$ ,  $A_2=1.$ ) and ( $F_1=.2$  ,  $F_2=.585$ ). Fifteen basis functions with frequency separation .05 were used. In this case, one of the frequencies was off the grid points, and subset size 8 were used. Frequency  $F_1$  was selected exactly along with frequencies around  $F_2$ . Note that frequency .6 was selected with a large magnitude which is the closest frequency from the set to  $F_2$ . The subsets examined were 1716 which are less than 2% per cent of the total possible subsets  $(15!/(8!7!) = 6435)$ . This example is illustrated in table 4.2 .

The third example was picked to show the improvement obtained by windowing tichnique. Thus, the amplitudes and frequencies were chosen to be the same as

example 2. In this case, the selected frequencies were .2 and .58437 after examining only 271 subsets. It was observed that the windowing technique offers better accuracy while the number of subsets examined are kept small. This example is illustrated in table 4.3 along with three more examples in table 4.4 to show the efficiency of this technique. Note that the results are very accurate, even for two tones with small frequency separation.

The next example is an illustration of frequency estimation in the presence of Gaussian noise. Five different signal-to-noise ratios (SNR) were used, and in each case the frequency estimates were averaged over 100 runs to evaluate the mean and the standard deviations. The results were tabulated in table 4.5 along with their corresponding C-R bounds. For high SNR's, the standard deviations of the estimates were consistant with the C-R bounds, but they differ for low SNR's due to the occurrence of the outliers. Outliers are extreme observations, and they create great difficulty. When we encounter one, our first suspicion is that the observation resulted from a mistake or other extraneous effect, and hence should be discarded. A major reason for discarding it is that under the least squares method, a fit curve is pulled disproportionately toward an outlying observation because of the squared diviations is minimized. This could cause a misleading fit if indeed the outlier obsevation resulted from a mistake or other extraneous cause. On the other

hand, outliers may convey significant information, as when an outlier occurs because of an interaction with another independent variable omitted from the model.

Two frequency estimation has recently been investigated by Kumaresan (15) in a different fashion. In his study, Kumaresan searches over the entire error surface to locate the minimum error, and therefore, the corresponding two frequencies. The same example has been run by his method and the results are listed in table 4.6 for comparison. Both methods give relatively identical results, and consistant with C-R bounds.

A small example with 10 runs is listed in table 4.7 to show how the output frequencies differ from the true values for each run. Two more examples are illustrated in tables 4.8 and 4.9 to show the estimation behavior of the two frequencies when the frequency separation is very small. The results in table 4.8 are consistant with the C-R bounds whereas the results in table 4.9 for stronger tone are lower than the C-R bounds due to the use of information about the tones magnitudes. That is, we have suplied more information than assumed in the C-R bounds for the estimation of the frequencies.

The C-R bounds are for unbiased estimation. We assume that our method, too, gives unbiased estimation since our results are consistant with C-R bounds, and try to show that this assumption is correct by analyzing the results.

We assume, as a working hypothesis, that the frequency error has a normal distribution with zero mean and standard deviation SDT. Then if our frequency error mean falls in the interval -SDT to SDT, we could say that the data is consistant with our zero-mean hypothesis to that accuracy. Since we have 100 independent runs, the probability density function of the frequency error is a chi-square distribution with 100 degrees of freedom which can be approximated accurately by a normal distibution (16) having the same mean and variance. Let us assume that our experimental standard deviation (SD) is in error by +20% from the true SDT (i.e.  $SDT = (1 \pm .2) SD$ ). And let us choose the worst case, namely  $SDT = (1 - .2) SD$ . If the frequency error mean is in the interval -SDT to SDT , we will say that with high probability it is an unbiased estimation. By checking table 4.5, we observe that even for low SNR , the frequency error mean falls in that interval. For instance, when SNR = 5db;

$$F1 - \hat{F}1 = .906E-3$$

$$SD1 = .772E-2$$

$$SDT = (1 - .2) SD1 = .6176E-2$$

$$SDT / (F1 - \hat{F}1) = 6.82$$

$$-SDT < F1 - \hat{F}1 < SDT$$

and therefore, our assumption is probably correct.

TABLE 4.1

NO. OF SAMMLES = 8  
 A1=0.1000D 01      A2=0.1000D 01  
 F1=0.2000D 00      F2=0.4000D 00  
 N=20  
 E( 3)=0.1703591D-13      NO. 1 SMALLEST ERROR  
 E( 4)=0.2899026D-12      NO. 2 SMALLEST ERROR  
 E( 2)=0.5724064D-12      NO. 3 SMALLEST ERROR  
 E( 5)=0.8619378D-12      NO. 4 SMALLEST ERROR  
 E( 6)=0.2004521D-11      NO. 5 SMALLEST ERROR  
 E( 1)=0.3271099D-11      NO. 6 SMALLEST ERROR  
 E( 7)=0.8995882D-11      NO. 7 SMALLEST ERROR  
 E( 8)=0.2952868D-10      NO. 8 SMALLEST ERROR  
 E( 9)=0.9718023D-10      NO. 9 SMALLEST ERROR  
 E(20)=0.1602580D-09      NO. 10 SMALLEST ERROR  
 E(10)=0.1609581D-09      NO. 11 SMALLEST ERROR  
 E(11)=0.1655543D-09      NO. 12 SMALLEST ERROR  
 E(12)=0.1662708D-09      NO. 13 SMALLEST ERROR  
 E(13)=0.1664244D-09      NO. 14 SMALLEST ERROR  
 E(14)=0.1664624D-09      NO. 15 SMALLEST ERROR  
 E(15)=0.1664727D-09      NO. 16 SMALLEST ERROR  
 E(16)=0.1664743D-09      NO. 17 SMALLEST ERROR  
 E(18)=0.1664748D-09      NO. 18 SMALLEST ERROR  
 E(19)=0.1664815D-09      NO. 19 SMALLEST ERROR  
 E(17)=0.1664869D-09      NO. 20 SMALLEST ERROR  
 N=20 M= 5  
 NO. OF SUBSETS EXAMINED = 136  
 D( 1)=0.2000D 00      C( 1)=(0.1000D 01 , 0.2664D-15 )  
 D( 2)=0.4000D 00      C( 2)=(0.1000D 01 , 0.1628D-14 )  
 D( 3)=0.8000D 00      C( 3)=(-.7860D-14 , -.7946D-14 )  
 D( 4)=0.8500D 00      C( 4)=(0.1166D-13 , -.1017D-13 )  
 D( 5)=0.9000D 00      C( 5)=(0.2896D-14 , 0.1049D-13 )  
 ERROR= 0.9073D-18

TABLE 4.2

NO. OF SAMPLES = 10

A1 = 0.10000D 01

A2 = 0.11000D 01

F1 = 0.20000D 00

F2 = 0.58500D 00

E( 8)=0.1150485D-05 NO. 1 SMALLEST ERROR

E(10)=0.1152479D-05 NO. 2 SMALLEST ERROR

E( 6)=0.1154274D-05 NO. 3 SMALLEST ERROR

E( 7)=0.1170815D-05 NO. 4 SMALLEST ERROR

E( 9)=0.1179624D-05 NO. 5 SMALLEST ERROR

E(13)=0.1238354D-05 NO. 6 SMALLEST ERROR

E( 1)=0.1276429D-05 NO. 7 SMALLEST ERROR

E(15)=0.1293429D-05 NO. 8 SMALLEST ERROR

E(14)=0.1298690D-05 NO. 9 SMALLEST ERROR

E( 2)=0.1327334D-05 NO. 10 SMALLEST ERROR

E( 5)=0.1353965D-05 NO. 11 SMALLEST ERROR

E( 3)=0.1435813D-05 NO. 12 SMALLEST ERROR

E(11)=0.1568995D-05 NO. 13 SMALLEST ERROR

E(12)=0.1861392D-05 NO. 14 SMALLEST ERROR

E( 4)=0.1992818D-05 NO. 15 SMALLEST ERROR

ERROR= 0.18303D-05

D( 1)=0.2000D 00 C = (0.99997D 00 -.24532D-03)

D( 2)=0.4000D 00 C = (-.44293D-02 0.77722D-02)

D( 3)=0.4500D 00 C = (-.36676D-01 -.29364D-01)

D( 4)=0.5000D 00 C = (0.10543D 00 -.95977D-01)

D( 5)=0.5500D 00 C = (0.22795D 00 0.34696D 00)

D( 6)=0.6000D 00 C = (0.69676D 00 -.31457D 00)

D( 7)=0.6500D 00 C = (0.22999D-01 0.84135D-01)

D( 8)=0.7000D 00 C = (-.12046D-01 0.13250D-02)

NO. OF SUBSETS EXAMINED = 1716

TABLE 4.3

A1=1.0            A2=1.0  
F1=.200           F2=.585

RUN	$\hat{F}_1$	$\hat{F}_2$	ERROR
1	0.2000	0.60000	.68822E-1
2	0.2000	0.57500	.31433E-1
3	0.2000	0.58750	.19749E-2
4	0.2000	0.58437	.12570E-3
5	0.2000	0.58437	.12570E-3

TOTAL SUBSETS EXAMINED = 271

TABLE 4.4

$A_1=1.0$        $A_2=1.0$   
 No. of samples = 10

	Case I F1=.2250 F2=.3125	Case II F1=.2500 F2=.7250	Case III F1=.1750 F2=.5875
D1	0.2000	0.3000	0.2000
D2	0.3000	0.7000	0.6000
Error	.2194E 0	.6768E 0	.2376E 0
WINDOW 1			
D1	0.2500	0.2500	0.1500
D2	0.3000	0.7500	0.6000
Error	.3485E-1	.1827E 0	.2037E 0
WINDOW 2			
D1	0.2250	0.2500	0.1750
D2	0.3000	0.7250	0.5750
Error	.2367E-1	.2000E-5	.4829E-1
WINDOW 3			
D1	0.2250	0.2500	0.1750
D2	0.3125	0.7250	0.5875
Error	.2000E-5	.2000E-5	.2000E-5
WINDOW 4			
D1	0.2250		0.1750
D2	0.3125	STOP	0.5875
Error	.2000E-5		.2000E-5
	STOP		STOP

TABLE 4.5

$A_1=1.$   
 $F_1=.25$   
 $N_{RUN}=100$

$A_2=1.$   
 $F_2=.375$   
No. of samples = 12

SNR	$\hat{F}_1$	$ F_1 - \hat{F}_1 $	SD1	$\hat{F}_2$	$ F_2 - \hat{F}_2 $	SD2
20	.24987	.125E-3	.136E-2	.37497	.314E-4	.148E-2
10	.24945	.547E-3	.424E-2	.37478	.219E-3	.456E-2
5	.24909	.906E-3	.772E-2	.37466	.375E-3	.828E-2
0	.24850	.150E-2	.484E-1	.40651	.315E-1	.128E-0
-5	.32156	.715E-1	.181E-0	.6109	.186E-0	.237E-0

SNR signal to noise ratio  
SD standard deviation  
 $\hat{F}_i$  mean estimate  
 $F_i - \hat{F}_i$  frequency error

TABLE 4.6

$A_1=1.$   
 $F_1=.25$   
 $N_{RUN}=100$

$A_2=1.$   
 $F_2=.375$   
No. of samples = 12

SNR	$ F_1 - \hat{F}_1 $	SD1	$ F_2 - \hat{F}_2 $	SD2	C-R BOUND
20	.161E-3	.128E-2	.460E-4	.146E-2	.133E-2
10	.250E-3	.573E-2	.230E-3	.550E-2	.420E-2
5	.230E-2	.470E-1	.290E-2	.050E-1	.748E-2
0	.339E-2	.589E-1	.338E-1	.124E 0	.133E-1
-5	.186E-2	.117E 0	.186E 0	.227E 0	.236E-1

TABLE 4.7

KN= 12 NRUN= 10  
 A1=0.1000D 01 A2=0.1000D 01  
 F1=0.2500D 00 F2=0.4750D 00  
 SIR = 0.0 SNR =0.1000E 02 NVAR=0.1000E 00  
 FREQ1 FREQ-ERROR1 FREQ2 FREQ-ERROR2  
 0.242187D 00 0.781259D-02 0.478125D 00 0.312483D-02  
 0.246875D 00 0.312509D-02 0.487500D 00 0.124998D-01  
 0.259375D 00 0.937491D-02 0.476562D 00 0.156233D-02  
 0.253125D 00 0.312491D-02 0.478125D 00 0.312483D-02  
 0.250000D 00 0.894070D-07 0.470312D 00 0.468767D-02  
 0.248437D 00 0.156259D-02 0.476562D 00 0.156233D-02  
 0.253125D 00 0.312491D-02 0.478125D 00 0.312483D-02  
 0.248437D 00 0.156259D-02 0.476562D 00 0.156233D-02  
 0.251562D 00 0.156241D-02 0.464062D 00 0.109377D-01  
 0.250000D 00 0.894070D-07 0.492187D 00 0.171873D-01  
 MEAN1 = 0.250312D 00 MEAN2 = 0.477812D 00  
 MERROR1=0.312410D-03 MERROR2=0.281233D-02  
 VARIANCE1=0.184570D-04 VARIANCE2=0.550781D-04  
 STDIV1 = 0.429616D-02 STDIV2 = 0.742146D-02

TABLE 4.8

$A_1=1.0$   
 $F_1=.5$   
 $NRUN=100$   
 $A_2=1.0$   
 $F_2=.55$   
 NO. of samples =12

SNR	$ F_1 - \hat{F}_1 $	SD1	$ F_2 - \hat{F}_2 $	SD2	C-R BOUND
20	.187E-3	.263E-2	.171E-3	.262E-2	.272E-2
10	.281E-3	.852E-2	.359E-3	.859E-2	.860E-2
5	.718E-3	.152E-1	.453E-3	.155E-1	.153E-1
0	.193E-1	.857E-1	.594E-1	.118E 0	.272E-1
-5	.420E-1	.167E 0	.321E-1	.179E 0	.484E-1

TABLE 4.9a

$A_1=1.$   
 $F_1=.25$   
 $N_{RUN}=100$

$A_2=10.$   
 $F_2=.275$   
No. of samples = 12

SNR	$\hat{F}_1$	$ F_1 - \hat{F}_1 $	SD1	$\hat{F}_2$	$ F_2 - \hat{F}_2 $	SD2
20	.25078	.781E-3	.643E-2	.27515	.156E-3	.781E-3
10	.25467	.467E-2	.224E-1	.27402	.984E-3	.264E-2
5	.26035	.103E-1	.428E-1	.27361	.139E-2	.342E-2
0	.27495	.249E-1	.725E-1	.27343	.156E-2	.391E-2
-5	.29096	.409E-1	.887E-1	.27309	.190E-2	.555E-2

TABLE 4.9b

$A_1=1.$   
 $F_1=.25$   
 $N_{RUN}=100$

$A_2=10.$   
 $F_2=.275$   
No. of samples = 12

SNR	C-R BD1	C-R BD2
20	.717E-2	.717E-3
10	.226E-1	.226E-2
5	.403E-1	.403E-2
0	.717E-1	.717E-2
-5	.127E+0	.127E-1

## CONCLUSION

The complex subset selection algorithm was described in chapter 2. This algorithm was used in section 3.2 to design low-pass FIR digital filters with a small number of coefficients using a weighted least squares error criterion. Digital low-pass filters with uniformly-spaced time samples were discussed in section 3.3, and it was seen in section 3.4 that, for the same time aperture and number of coefficients, the complex subset selection method results in a) smaller least squares error, b) flatter magnitude response in the passband region, c) sharper roll-off in the transition band, and d) higher peak sidelobe level.

Tone estimation by the complex subset selection method was explained in section 4.2, and was extended to frequency estimation in section 4.3. The presence of Gaussian noise was discussed in section 4.4. The numerical results were compared with the C-R bounds in section 4.5, where unbiased estimation was assumed, and it was shown that the results were consistent with the C-R bounds. Finally, it should be mentioned that this method can be applied to multi-frequency estimation with a little modification, and with some more computation.

## APPENDIX A

In this appendix, the details of the derivation of digital filter synthesis is presented.

$$D(w) = \begin{cases} 1 & , w < p \\ 0 & , s_1 < w < s_2 \\ \text{Don't care} & , \text{otherwise} \end{cases} \quad (\text{A.1})$$

$$W(w) = \begin{cases} 1 & , |w| \leq p \text{ and } s_1 \leq |w| \leq s_2 \\ 0 & , \text{otherwise} \end{cases} \quad (\text{A.2})$$

$$H(w) = \sum_{n=1}^M h(n) \cdot \text{EXP}(JwT_n) \quad (\text{A.3})$$

Where  $T_n$ 's are selected from a set  $\{nT\}$  with  $n=-q, \dots, q$ .

$$\text{Min. Error} = \int_{\{h(n), T_n\}} W(w) \left| D(w) - H(w) \right|^2 dw \quad (\text{A.4})$$

Using orthogonality principle (14), it follows:

$$\int W(w) \cdot (D(w) - H(w)) \cdot \text{EXP}(-JwT_k) \cdot dw = 0 \quad \text{for } k=1, \dots, M \quad (\text{A.5})$$

$$\int W(w) \cdot H(w) \cdot \text{EXP}(-JwT_k) \cdot dw = \int W(w) \cdot D(w) \cdot \text{EXP}(-JwT_k) \cdot dw$$

for  $k=1, \dots, M$  (A.6)

$$A \cdot H = B \quad (A.7)$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1M} \\ a_{21} & & & \\ a_{M1} & \dots & a_{MM} \end{bmatrix} \quad (A.8)$$

$$H = [h(n), \dots, h(M)] \quad (A.9)$$

$$B = [b, \dots, b] \quad (A.10)$$

and

$$\begin{aligned} a_{ij} &= \int W(w) \cdot \text{EXP}(Jw(T_i - T_j)) \cdot dw = \int_{S1}^{S1} \text{EXP}(Jw(T_i - T_j)) \cdot dw \\ &\quad + \int_{-P}^P \text{EXP}(Jw(T_i - T_j)) \cdot dw + \int_{S2}^{S2} \text{EXP}(Jw(T_i - T_j)) \cdot dw \\ &= (\text{EXP}(-Js1(T_i - T_j)) - \text{EXP}(-Js2(T_i - T_j))) / J(T_i - T_j) \\ &\quad + (\text{EXP}(Jp(T_i - T_j)) - \text{EXP}(-Jp(T_i - T_j))) / J(T_i - T_j) \\ &\quad + (\text{EXP}(Js2(T_i - T_j)) - \text{EXP}(Js1(T_i - T_j))) / J(T_i - T_j) \\ &= 2 [\sin(s2(T_i - T_j)) - \sin(s1(T_i - T_j))] \\ &\quad + \sin(p(T_i - T_j)) \pi / (T_i - T_j) \quad (A.11) \end{aligned}$$

$$b_i = \int W(w) \cdot D(w) \cdot \text{EXP}(-JwT_i) \cdot dw = \int_{-P}^P \text{EXP}(-JwT_i) \cdot dw$$

$$= \left[ \text{EXP}(-JpT_i) - \text{EXP}(JpT_i) \right] / (-JT_i) = 2 \sin(pT_i) / T_i$$

(A. 12)

Therefore,

$$H = A \cdot B \quad (A. 13)$$

$$\begin{aligned} \text{Min. Error} &= \int_{h(n), T} W(w) \left| D(w) - \sum_{n=1}^M h(n) \cdot \text{EXP}(JwT_n) \right|^2 dw \\ &= \int_{h(n), T} W(w) (D(w) - \sum_{n=1}^M h(n) \cdot \text{EXP}(JwT_n)) \cdot D(w) \cdot dw \end{aligned}$$

(A. 14)

$$\begin{aligned} \text{Min. Error} &= \int_{\{h(n), T_n\}} W(w) |D(w)|^2 dw \\ &= \sum_{n=1}^M h(n) \int_{W(w) \cdot D(w) \cdot \text{EXP}(JwT_n)} dw \\ &= \int_{-P}^P 1 \cdot dw - \sum_{n=1}^M h(n) \cdot \text{EXP}(JwT_n) \cdot dw \\ &= 2p - \sum_{n=1}^M h(n) \cdot b_n \end{aligned}$$

(A. 15)

## **APPENDIX R**

FOUR DIFFERENT PROGRAMS ARE LISTED BELOW WHICH EACH ONE IS FOR AN ESPECIAL PURPOSE. PROGRAM ONE AND TWO ARE FOR DIGITAL FILTER DESIGN WHILE PROGRAMS 3 AND 4 ARE FOR TONE AND FREQUENCY ESTIMATION.

## **PROGRAM-1**

THIS IS THE PROGRAM LISTING FOR DIGITAL FILTER DESIGN BY COMPLEX SUBSET SELECTION.

A(I,J)	A MATRIX
COL(I)	B MATRIX
SOL(I)	COEFFICIENT MATRIX
E(I)	EI ERROR FUNCTIONS
AE(I)	RE-ORDER ERRORS
TN(T)	BASIS FUNCTION PARAMETERS
NQ	Q IN EQUATION (3.2.1)
M	NO. OF COEFFICIENTS
P	PASSBAND EDGE
S1	LEFT STOPBAND EDGE
S2	RIGHT STOPBAND EDGE
T	GRID POINT SEPARATION

```

DIMENSION Y(101)
COMPLEX U,ER,H(30),ET,BRSS
REAL E(30),AE(30),TN(30),CT(10)
INTEGER IP(30),IND(30)
INTEGER IP1(30),IP2(30)
COMPLEX A(30,30),A1(30,30),ICOL(30),COL(30)
COMPLEX SOL(30),SOL1(30)
DATA A,COL,A1/900*(0.,0.),30*0.0,900*(0.,0.)/
DATA IND,ICOL/30*0,30*0.0/
READ(5,210)NQ,M,NO,F,S1,S2,T
READ(5,220)MARK,NW,NP
PI=4.0*ATAN(1.0)
NQ1=2*NQ+1
IF(MARK.EQ.1)GO TO 5
DO 1 I=1,NQ1
TN(I)=(I-NQ-1)*T

```

```

1 CONTINUE
GO TO 8
5 READ(5,230)(CT(I),I=1,NW)
TH=T/2
DO 7 I=1,NP
DO 7 J=1,NW
TN(I+(J-1)*NP)=CT(J)+(I-2)*TH
7 CONTINUE
NQ1=NP*NW
8 WRITE(6,150)NQ,M
WRITE(6,240)P,S1,S2,T
P=P*2*PI
S1=S1*2*PI
S2=S2*2*PI

C
C EVALUATION OF THE ELEMENTS OF A AND B MATRIX,
C EQUATIONS (3.2.14) AND (3.2.15)
C
DO 32 I=1,NQ1
DO 20 J=1,NQ1
TIJ=TN(I)-TN(J)
IF(TIJ.EQ.0.)GO TO 10
A(I,J)=2*(SIN(TIJ*S2)-SIN(TIJ*S1)+SIN(TIJ*P))/TIJ
A(J,I)=A(I,J)
GO TO 20
10 A(I,J)=2*(S2-S1+P)+.0001
20 CONTINUE
IF(TN(I).EQ.0.)GO TO 30
COL(I)=2*SIN(P*TN(I))/TN(I)
GO TO 32
30 COL(I)=2*P
32 CONTINUE
CALL ANERIV(A,COL,H,ET1,NQ1)
ET1=2*P-ET1
WRITE(6,190)ET1

C
C EVALUATION AND PLOTTING OF THE FREQUENCY RESPONSE
C FOR THE FILTER USING ALL THE BASIS FUNCTIONS
C
NQ1=NQ+1
L=0
DO 38 I=1,NQ1
V=CMPLX(0.,0.)
DO 37 J=1,NQ1
X=2*PI*L*TN(J)/NQ
V=V+H(J)*CEXP(CMPLX(0.,X))
37 CONTINUE
L=L+1
Y(I)=CABS(V)
Y(I)=20*ALOG(Y(I)+.0000001)

```

```

38 CONTINUE
RX=1./NO
CALL FPLOTT(Y,NQ1,0.,DX,2.5,-100.0)
C
C EVALUATION AND RE-ORDERING OF EI ERROR FUNCTIONS,
C EQUATIONS (2.2.9) AND (2.2.10)
C
NS=NQ1-1
DO 40 IE=1,NQ1
CALL DELT(A,COL,A1,DCOL,NQ1,NS,IE)
CALL AHERIV(A1,DCOL,SOL,ET1,NS)
ET1=2*P-ET1
E(IE)=ET1
40 CONTINUE
DO 60 K=1,NQ1
AE(K)=E(K)
IP(K)=K
DO 50 LP=1,NQ1
IF(AE(K).LE.E(LP))GO TO 50
AE(K)=E(LP)
IP(K)=LP
50 CONTINUE
E(IP(K))=1.0E 6
WRITE(6,170)IP(K),AE(K),K
60 CONTINUE
C
C BEGINING OF THE SEARCH FOR BEST SUBSET
C
NR=NQ1-M
DO 70 I=1, NR
IND(I)=1
70 CONTINUE
DO 80 J=1,M
IND(J+NR)=0
80 CONTINUE
IE=-1
MF=NR+1
RR=AE(MF)
KP=1
KPT=1
MI=M+1
ETAN=1.0E 6
DO 90 IT=1, MI
IF(IT.EQ.1)GO TO 82
KP=(KP*(NR+IT-2))/(IT-1)
KPT=KPT+KP
82 DO 86 K=1,KP
IF(IT.EQ.1)GO TO 83
CALL COMBO(IND,NQ1, NR, NM)
83 CALL INTG(A,COL,A1,DCOL,H,IND,IP,IP1,NQ1,TN,NS,IE)

```

```

CALL AHERIV(A1,DCOL,SOL,ET1,NS)
ET1=2*P-ET1
ETA=ET1
IF(ETA.GT.ETAN)GO TO 86
ETAN=ETA
DO 85 IX=1,M
SOL1(IX)=SOL(IX)
IP2(IX)=IP1(IX)
85 CONTINUE
86 CONTINUE
IF(ETAN.LE.BR)GO TO 92
MP=MP+1
BR=AE(MP)
90 CONTINUE
92 DO 93 L=1,M
WRITE(6,180)L,SOL1(L),TN(IP2(L))
93 CONTINUE
WRITE(6,190)ETAN
WRITE(6,200)KPT
C
C EVALUATION AND PLOTTING OF THE FREQUENCY RESPONSE
C FOR A BEST SUBSET SELECTED FILTER
C
L=0
DO 94 I=1,N01
V=CMPLX(0.,0.)
DO 95 J=1,M
X=2*PI*L*TN(IP2(J))/N0
V=V+SOL1(J)*CEXP(CMPLX(0.,X))
95 CONTINUE
L=L+1
Y(I)=CABS(V)
Y(I)=20*ALOG(Y(I)+.0000001)
94 CONTINUE
CALL FFLOTT(Y,N01,0.,DX,2.5,-100.0)
100 FORMAT(2I2,F10.0)
110 FORMAT(4F10.0)
120 FORMAT(5X,'D('',I2,'')=',E10.2)
130 FORMAT(//,5X,'C('',I2,'')=( ',E12.4,' , ',E12.4,' )')
140 FORMAT(//,5X,'RSS FOR FULL BASIS =',E12.4)
150 FORMAT(//,5X,'N=',I2,' M=',I2)
160 FORMAT(//,5X,'E('',I2,'')=',E12.4)
170 FORMAT(//,5X,'E('',I2,'')=',E14.7,' NO. ',I2,
*' SMALLEST ERROR')
180 FORMAT(//,' H('',I2,'')=( ',E12.4,' , ',E12.4,' )',
*' T=',E10.4)
190 FORMAT(//,5X,'ERROR=',E14.7)
200 FORMAT(//,5X,'NO. OF SUBSETS EXAMINED ',I6)
210 FORMAT(3I3,4F5.0)
220 FORMAT(I1,2I2)

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```

230 FORMAT(5F5.0)
240 FORMAT(/,5X,'PBAND =',E10.4,3X,'SBAND1 =',E10.4,3X,
*'SBAND2 =',E10.4,3X,'T=',E10.4)
      STOP
      END

C
C      SUBROUTINE INTG IS FOR SELECTING THE ELEMENTS
C      OF A AND B MATRICES FOR THE INDICATED SUBSET
C
      SUBROUTINE INTG(A,COL,A1,DCOL,H,IND,IP,IP1,NQ1,TN,NS,IE)
      COMPLEX A(30,30),COL(30),A1(30,30),DCOL(30),H(30)
      REAL TN(30)
      INTEGER IND(30),IP(30),IP1(30)

25   K=0
      K1=1
      DO 26 K2=1,NQ1
      IF(IND(K2).EQ.1)GO TO 26
      IP1(K1)=IP(K2)
      K=K+1
      K1=K1+1
26   CONTINUE
      NS=K
      DO 2 I=1,NS
      DO 2 J=I,NS
      A1(I,J)=A(IP1(I),IP1(J))
      A1(J,I)=A1(I,J)
2     CONTINUE
      DO 3 K=1,NS
      DCOL(K)=COL(IP1(K))
3     CONTINUE
      RETURN
      END

C
C      SUBROUTINE AHERIV IS FOR SOLVING THE LINEAR
C      EQUATIONS AH=B, AND EVALUATING THE
C      APPROPRIATE ERROR
C
      SUBROUTINE AHERIV(A,COL,SOL,ET1,K)
      COMPLEX A(30,30),COL(30),RL(30,30),SOL(30),E1,E2,E3,ET
      DO 5 I=1,K
      DO 5 J=1,K
5     RL(I,J)=CMPLX(0.,0.)
      DO 60 I=1,K
      IP1=I+1
      IM1=I-1
      IF(I.NE.1)GO TO 30
      DO 10 J=1,K
10     RL(1,J)=A(1,J)

```

```

DO 20 J=2,K
20 RL(J,1)=CONJG(RL(1,J))/RL(1,1)
GO TO 60
30 DO 40 J=I,K
RL(I,J)=A(I,J)
DO 40 M=1,IM1
40 RL(I,J)=RL(I,J)-RL(I,M)*RL(M,J)
IF(I.EQ.K)GO TO 60
DO 45 J=IP1,K
45 RL(J,I)=CONJG(RL(I,J))/RL(I,I)
60 CONTINUE
80 DO 100 I=1,K
IM1=I-1
SOL(I)=COL(I)
IF(I.EQ.1)GO TO 100
DO 90 L=1,IM1
90 SOL(I)=SOL(I)-SOL(L)*RL(I,L)
100 CONTINUE
DO 140 I=1,K
J=K-I+1
IF(J.EQ.K)GO TO 130
JP1=J+1
DO 120 L=JP1,K
120 SOL(J)=SOL(J)-SOL(L)*RL(J,L)
130 SOL(J)=SOL(J)/RL(J,J)
140 CONTINUE
C
C      EVALUATION OF ERROR, EQUATION (3.2.17)
C
ET=0.
DO 150 I=1,K
ET=ET+COL(I)*SOL(I)
150 CONTINUE
ET1=CABS(ET)
RETURN
END
C
C      SUBROUTINE COMBO GENERATES STRING OF ONES AND
C      ZEROS FOR DELETION OF BASIS FUNCTIONS
C
SUBROUTINE COMBO(JN,NQ1,M,NF)
DIMENSION JN(30)
ONES=M
NF=0
1 NF=1
2 CONTINUE
IF(JN(NF).EQ.0)GO TO 10
IF(NF.EQ.NQ1)GO TO 101
JN(NF)=0

```

```

NP=NP+1
ONES=ONES-1
GO TO 2
10 JN(NP)=1
ONES=ONES+1
IF(ONES.EQ.M)GO TO 100
GO TO 1
101 NF=1
100 RETURN
END
C
C
C      SUBROUTINE DELT IS FOR DELETING ROW AND
C      COLUMN OF A AND B MATRICES FOR EVALUATION
C      OF EI ERROR FUNCTIONS
C
SUBROUTINE DELT(A,COL,A1,DCOL,NQ1,NS,IE)
COMPLEX A(30,30),COL(30),A1(30,30),DCOL(30)
IEM=IE-1
IF(IE.EQ.1)GO TO 30
DO 10 I=1,IEM
DCOL(I)=COL(I)
DO 10 J=I,IEM
A1(I,J)=A(I,J)
A1(J,I)=A(J,I)
10 CONTINUE
DO 20 I=1,IEM
DO 20 J=IE,NS
A1(I,J)=A(I,J+1)
A1(J,I)=A(J+1,I)
20 CONTINUE
IF(IE.EQ.NQ1)GO TO 50
30 DO 40 I=IE,NS
DCOL(I)=COL(I+1)
DO 40 J=I,NS
A1(I,J)=A(I+1,J+1)
A1(J,I)=A(J+1,I+1)
40 CONTINUE
50 RETURN
END
C
C
C      SUBROUTINE FPLOTT IS FOR PLOTTING THE
C      FREQUENCY RESPONSE OF THE FILTER
C
SUBROUTINE FPLOTT(Y,NDIM,XI,DX,YMAX,YMIN)
DIMENSION Y(NDIM)
DATA IBLANK,IPOINT,IAST,IM1NUS// ' ',.,*,/-/,IPRINT/6/
INTEGER PLOT(78)
IF(YMAX-YMIN)>250,5,45

```

```

5      YMAX=Y(1)
     YMIN=Y(1)
     DO 40 K=1,NDIM
        IF(Y(K)-YMAX)20,20,10
10     YMAX=Y(K)
20     IF(Y(K)-YMIN)30,40,40
30     YMIN=Y(K)
40     CONTINUE
        IF(YMAX-YMIN)250,250,45
45     X=XI-DX
        IF(YMIN)60,60,50
50     IQ=0
     GO TO 70
60     IF(YMAX)62,68,68
62     IQ=76
     GO TO 70
68     IQ=IFIX(76.*(-YMIN/(YMAX-YMIN))+.5)
70     KI=IQ+1
        WRITE(IPRINT,1000)YMIN
1000   FORMAT(' YMIN= ',1PE10.3)
        WRITE(IPRINT,2000)YMAX
2000   FORMAT(' YMAX= ',1PE10.3/)
        DO 200 K=1,NDIM
           YK=Y(K)
           IF(YK-YMAX)81,84,80
80     YK=YMAX
     GO TO 83
81     IF(YK-YMIN)82,84,84
82     YK=YMIN
83     IQUE=IAST
     GO TO 85
84     IQUE=IBLANK
85     CONTINUE
        IP=IFIX(76.*((YK-YMIN)/(YMAX-YMIN))+.5)
        IR=IP+1
        X=X+DX
        IF(YK)90,150,130
90     DO 100 I=1,IP
100    PLOT(I)=IBLANK
        PLOT(IP)=IAST
        J=IP+1
        DO 110 L=J,IQ
110    PLOT(L)=IMINUS
        IF(IP-IQ)130,120,130
120    PLOT(KI)=IAST
     GO TO 140
130    PLOT(KI)=IPOINT
140    WRITE(IPRINT,3000)X,Y(K),IQUE,(PLOT(M),M=1,KI)
     GO TO 200
150    DO 160 I=1,KI

```

```
160 PLOT(I)=IBLANK
      PLOT(KI)=IPOINT
      J=KI+1
      DO 170 I=J,IP
170 PLOT(I)=IMINUS
      PLOT(IR)=IAST
      WRITE(IPRINT,3000)X,Y(K),IQUE,(PLOT(M),M=1,IR)
3000 FORMAT(' X=',1PE10.3,3X,' Y=',1PE10.3,3X,A1,' \',77A1)
200 CONTINUE
      RETURN
250 WRITE(IPRINT,5000)
5000 FORMAT(' ERROR--YMAX=YMIN')
      RETURN
      END
```

```
C PROGRAM-2
C
C
C THIS IS THE PROGRAM LISTING FOR UNIFORMLY-
C SPACE PARAMETER FILTER
C
C EVALUATION OF THE ELEMENTS OF A' AND B' MATRICES
C OF EQUATION (3.3.3)
C
C A(I,J)      A' MATRIX
C B(I)        B' MATRIX
C H(I)        COEFFICIENT MATRIX
C M           NO. OF COEFFICIENTS
C P           PASSBAND EDGE
C S1          LEFT STOPBAND EDGE
C S2          RIGHT STOPBAND EDGE
C ND          NO. OF DELTAS FOR SEARCH
C NC          NO. OF C'S FOR SEARCH
C DEL         INITIAL DELTA
C DELD        DELTA INCREMENT
C CI          INITIAL C
C DELC        C INCREMENT
C
C DIMENSION DN(10),Y(101),E(100)
C COMPLEX H(10),A(10,10),B(10),HR(10),ET,V
C READ(5,100)NO,M,ND,NC
C READ(5,110)F,S1,S2,DEL,DELD
C READ(5,110)CI,DELC
C PI=4*ATAN(1.)
C D1=DEL
C P=2*PI*F
C S1=2*PI*S1
C S2=2*PI*S2
C SER=1.0E 6
C K1=4
C CO=CI
C DO 50 IK=1,NC
C DO 40 IL=1,ND
C DO 25 I=1,M
C DO 15 J=I,M
C IF(I.EQ.J)GO TO 10
C X=D1*(I-J)
C A(I,J)=2*(SIN(S2*X)-SIN(S1*X)+SIN(P*X))/X
C A(J,I)=A(I,J)
```

```

      GO TO 15
10   A(I,J)=2*(S2-S1+F)+.0001
15   CONTINUE
     X=(I-3)*D1+C0
     IF(X.EQ.0.)GO TO 20
     B(I)=2*SIN(F*X)/X
     GO TO 25
20   B(I)=2*P
25   CONTINUE
     CALL AHERIV(A,B,H,E1,ET,M)

C   SELECTION OF THE SMALLEST ERROR AND
C   CORRESPONDING COEFFICIENTS
C
     E(IL)=E1
     ET1=2*K-P-E1
     WRITE(6,125)C0,D1,ET1
     IF(SER.LT.ET1)GO TO 35
     SER=ET1
     CB=C0
     BD=D1
     DO 30 L=1,M
30   HB(L)=H(L)
35   D1=D1+DELD
40   CONTINUE
     CO=C0+DELC
     D1=DEL
50   CONTINUE
     WRITE(6,130)CB,BD,SER
     DO 60 I=1,M
60   WRITE(6,140)HB(I)
     DO 62 I=1,M
62   DN(I)=(I-3)*BD+CB
     WRITE(6,150)(DN(I),I=1,M)

C   EVALUATION AND PLOTTING OF FREQUENCY RESPONSE
C
     NO1=NO+1
     L=0
     DO 70 I=1,NO1
     V=CMPLX(0.,0.)
     DO 65 J=1,M
     X=2*PI*L*DN(J)/NO
     V=V+HB(J)*CEXP(CMPLX(0.,X))
65   CONTINUE
     L=L+1
     Y(I)=CABS(V)
     Y(I)=20* ALOG(Y(I)+.0000001)
70   CONTINUE
     DX=1./NO

```

```

      CALL FPLOTT(Y,NO1,0.,DX,4.,-100.0)
      CALL FPLOTT(E,ND,DEL,DELC,1.8,0.0)
100  FORMAT(I3,3I2)
110  FORMAT(5F5.0)
120  FORMAT(/,5X,' DELTA=',E10.4,' ERROR=',3E12.5)
125  FORMAT(3X,'C=',E9.3,3X,'DELTA=',E8.2,3X,'ERROR=',E12.6)
130  FORMAT(/,5X,'BEST C=',E10.4,' BEST DELTA=',E10.4,
     *' ERROR=',2E12.5)
140  FORMAT(/,5X,'HB =',2E12.5)
150  FORMAT(/,5X,5(E10.4,2X))
      STOP
      END
C
C
C      SUBROUTINE AHERIV IS FOR SOLVING THE LINEAR
C      EQUATIONS A'H=B', AND EVALUATING THE
C      CORRESPONDING ERROR FUNCTION
C
C      SUBROUTINE AHERIV(A,COL,SOL,E1,ET,K)
COMPLEX A(10,10),COL(10),RL(10,10),SOL(10),E1,E2,E3,ET
DO 5 I=1,K
DO 5 J=1,K
5   RL(I,J)=CMPLX(0.,0.)
DO 60 I=1,K
IF1=I+1
IM1=I-1
IF(I.NE.1)GO TO 30
DO 10 J=1,K
10   RL(I,J)=A(I,J)
DO 20 J=2,K
20   RL(J,1)=CONJG(RL(1,J))/RL(1,1)
      GO TO 60
30   DO 40 J=I,K
      RL(I,J)=A(I,J)
      DO 40 M=1,IM1
40   RL(I,J)=RL(I,J)-RL(I,M)*RL(M,J)
      IF(I.EQ.K)GO TO 60
      DO 45 J=IF1,K
45   RL(J,I)=CONJG(RL(I,J))/RL(I,I)
60   CONTINUE
80   DO 100 I=1,K
      IM1=I-1
      SOL(I)=COL(I)
      IF(I.EQ.1)GO TO 100
      DO 90 L=1,IM1
90   SOL(I)=SOL(I)-SOL(L)*RL(I,L)
100  CONTINUE
      DO 140 I=1,K
      J=K-I+1
      IF(J.EQ.K)GO TO 130

```

```
IF(J.EQ.K)GO TO 130
JP1=J+1
DO 120 L=JP1,K
120 SOL(J)=SOL(J)-SOL(L)*RL(J,L)
130 SOL(J)=SOL(J)/RL(J,J)
140 CONTINUE
C
C      EVALUATION OF ERROR, EQUATION (3.3.4)
C
ET=0.
DO 150 I=1,K
ET=ET+COL(I)*SOL(I)
150 CONTINUE
E1=CABS(ET)
RETURN
END
```

```

C      PROGRAM-3
C
C
C      THIS IS THE PROGRAM LISTING FOR
C      TONE ESTIMATION.
C
C      A1,A2      TONES AMPLITUDES
C      B1,B2      TONES FREQUENCIES
C      KN         NO. OF SAMPLES
C      A(I,J)     A MATRIX
C      COL(I)    B MATRIX
C      SOL(I)    COEFFICIENT MATRIX
C      E(I)      EI'S ERROR FUNCTIONS
C      SIR       SIGNAL TO INTERFERENCE RATIO
C      SNR       SIGNAL TO NOISE RATIO
C
COMMON PI,A1,A2,B1,B2,R1,R2,D3,D4,N1,N2,KN
REAL*8 D(50),E(50),AE(50),A1,A2,B1,B2,PI,D1,D2
REAL*8 D3,D4,P1,P2,EC3,P3,BR,ETA,ETAN
INTEGER IF(50),IND(50),INDR(50),INP(50),JP(50)
INTEGER JJP(50),LJP(50),R
COMPLEX*16 A(50,50),COL(50),SOL(50),SOL1(50),ET
COMPLEX*16 S(50,50),DCOL(50),BRSS,Y(100),VN,E3
DATA A,COL,IND/2500*(0.0D0,0.0D0),50*(0.0D0,0.0D0),50*0/
PI=4.0D0*DATAN(1.0D0)
READ(5,110)SNR,SIR,P1,R2
IND1=0
IND3=0
A1=1.D0
A2=A1*(10**(-SIR/20))
READ(5,105)KN,AK
IX=3657137
SD=A1*(10**(-SNR/20))
VAR=SD**2
WRITE(6,113)KN
WRITE(6,115)A1,A2
WRITE(6,117)B1,B2
10 READ(5,100)LFI,N,MS,ML,D1,D2
IF(LFI.EQ.1)GO TO 2
C
C      GENERATION OF BASIS FUNCTION PARAMETERS
C
D(1)=D1
DO 1 I=2,N
1  D(I)=D(I-1)+D2
GO TO 5
2  READ(5,220)N1,N2,D3,D4

```

```

      D(1)=D1
      DO 3 I=2,N1
 3   D(I)=D(I-1)+D2
      D(N1+1)=D3
      DO 4 J=2,N2
 4   D(N1+J)=D(N1+J-1)+D4
      N=N1+N2
 5   WRITE(6,210)N
      IF(IND1.EQ.1)GO TO 11
C
C   GENARATION OF GAUSSIAN NOISE IN DATA,
C   EQUATIONS (4.2.2) AND (4.4.1)
C
      DO 7 I=1,KN
      CALL GAUSS(IX,SD,0.,V)
      CALL GAUSS(IX,SD,0.,V1)
      VN=CMPLX(V,V1)
      P1=2.D0*PI*B1
      P2=2.D0*PI*B2
      Y(I)=A1*CDEXP(DCMPLX(0.D0,P1*(I-1)))
      Y(I)=Y(I)+A2*CDEXP(DCMPLX(0.D0,P2*(I-1)))
 7   CONTINUE
 11  IF(IND3.EQ.0)GO TO 8
      P1=2.*PI*D(LJP(1))
      P2=2.*PI*D(LJP(2))
      DO 9 I=1,KN
      Y(I)=Y(I)-SOL1(1)*CDEXP(DCMPLX(0.D0,P1*(I-1)))
      Y(I)=Y(I)-SOL1(2)*CDEXP(DCMPLX(0.D0,P2*(I-1)))
 9   CONTINUE
 8   NS=N
C
C   EVALUATION AND RE-ORDERING OF ET ERROR
C   FUNCTIONS, EQUATIONS (2.2.9) AND (2.2.10)
C
      IE=0
      CALL INTG(A,COL,Y,IND,IP,N,NS,LFI,IE)
      CALL AHERIV(A,COL,SOL,Y,ET,NS,KN)
      ETA=CDARS(ET)
      NS=N-1
      DO 40 IE=1,N
      CALL DELT(A,COL,S,DCOL,N,NS,IE)
      CALL AHERIV(S,DCOL,SOL,Y,ET,NS,KN)
      E(IE)=CDARS(ET)
 40  CONTINUE
      DO 60 K=1,N
      AE(K)=E(K)
      IP(K)=K
      DO 50 LP=1,N
      IF(AE(K).LE.E(LP))GO TO 50
      AE(K)=E(LP)

```

```
      IP(K)=LP
50  CONTINUE
     E(IP(K))=1.0E 6
60  CONTINUE
     DO 2000 IXX=1,N
        WRITE(6,170)IP(IXX),AE(IXX),IXX
2000 CONTINUE
C
C      BEGINING OF SEARCH FOR THE BEST SUBSET
C
     DO 99 M=MS,ML
        WRITE(6,150)N,M
        NR=N-M
        DO 70 I=1,NR
           IND(I)=1
           INDR(I)=1
70  CONTINUE
        DO 80 J=1,M
           IND(J+NR)=0
           INDR(J+NR)=0
80  CONTINUE
        IE=-1
        MP=NR+1
        BR=AE(MP)
        KP=1
        KPT=1
        MI=M+1
        ETAN=1.0E 6
        DO 90 IT=1,MI
           IF(IT.EQ.1)GO TO 82
           KP=(KP*(NR+IT-2))/(IT-1)
           KPT=KPT+KP
82  DO 86 K=1,KP
           IF(IT.EQ.1)GO TO 83
           CALL COMBO(IND,N,NR,NF)
83  CALL INTG(A,COL,Y,IND,IP,N,NS,LFI,IE)
           CALL AHERIY(A,COL,SOL,Y,ET,NS,KN)
           ETA=CDARS(ET)
           IF(ETA.GT.ETAN)GO TO 86
           ETAN=ETA
           ETAN=ETA
           DO 84 I=1,N
              INDR(I)=IND(I)
84  CONTINUE
           DO 85 J=1,M
              SOL1(J)=SOL(J)
85  CONTINUE
86  CONTINUE
87  IF(ETAN.LE.BR)GO TO 92
88  MP=MP+1
```

```

      BR=AE(MP)
90  CONTINUE
92  J=0
    DO 93 L=1,N
    IF(INDR(L).EQ.1)GO TO 93
    J=J+1
    JP(J)=IP(L)
93  CONTINUE
    DO 95 LP=1,J
    LJF(LP)=JP(LP)
    JJF(LP)=LP
    DO 94 KP=1,J
    IF(LJF(LP).LE.JP(KP))GO TO 94
    LJF(LP)=JP(KP)
    JJF(LP)=KP
94  CONTINUE
    JP(JJF(LP))=1E 6
95  CONTINUE
    WRITE(6,200)KPT
    DO 96 L=1,J
    WRITE(6,180)L,B(LJF(L)),L,SOL1(L)
96  CONTINUE
    WRITE(6,190)ETAN
99  CONTINUE
    READ(5,240)IND2,IND1,IND3
    IF(IND3.EQ.1)GO TO 10
    IF(IND2.EQ.1)GO TO 101
    GO TO 10
101 CONTINUE
100 FORMAT(I1,3I2,2F10.0)
105 FORMAT(I3,F10.0)
110 FORMAT(4F10.0)
113 FORMAT(/,5X,'NO. OF SAMMLES =',I4)
115 FORMAT(/,5X,'A1=''E10.4,5X,'A2=''E10.4)
117 FORMAT(/,5X,'F1=''E10.4,5X,'F2=''E10.4)
120 FORMAT(5X,'B('',I2,'')='',E12.4)
125 FORMAT(/,5X,'SIR='',E10.4,5X,'SNR='',E10.4,5X,
*'VAR='',E10.4)
130 FORMAT(/,5X,'C('',I2,'')= ('',E12.4,' ',',E12.4,' ')')
140 FORMAT(//,5X,'RSS FOR FULL BASIS ='',E12.4)
150 FORMAT(//,5X,'N='',I2,' M='',I2)
160 FORMAT(//,5X,'E('',I2,'')='',F13.7)
170 FORMAT(//,5X,'E('',I2,'')='',E13.7,' NO. '',I2,
*' SMALLEST ERROR')
180 FORMAT(//,5X,'B('',I2,'')='',E10.4,5X,'C('',I2,'')= ('',
*E.10.4,' ',',E10.4,' ')')
190 FORMAT(//,5X,'ERROR='',E12.4)
200 FORMAT(//,5X,'NO. OF SUBSETS EXAMINED = ',I4)
210 FORMAT(//,5X,'N='',I2)
220 FORMAT(2I2,2F10.0)

```

```

230 FORMAT(//,5X,'SUM OF COEFFICIENTS SQUARED =',E12.4)
240 FORMAT(3I1)
STOP
END
C
C      SUBROUTINE INTG IS FOR EVALUATION OF ELEMENTS OF
C      A AND B MATRICES FOR EACH SUBSET TO BE EXAMINED,
C      EQUATIONS (4.2.14) THRU (4.2.16)
C
C      SUBROUTINE INTG(A,COL,Y,IND,IP,N,NS,LFI,IE)
COMMON PI,A1,A2,B1,B2,D1,D2,D3,D4,N1,N2,KN
COMPLEX*16 A(50,50),COL(50),Y(50)
REAL*8 D(50),AD(50),X,Y1,Y2,Y3,A1,A2,B1,B2,PI
REAL*8 D1,D2,D3,D4,AK
INTEGER IND(50),IP(50),JP(50)
IF(LFI.EQ.1)GO TO 2
D(1)=D1
DO 1 L=2,N
1   D(L)=D(L-1)+D2
    GO TO 5
2   D(1)=D1
    DO 3 L=2,N1
3   D(L)=D(L-1)+D2
    D(N1+1)=D3
    DO 4 LL=2,N2
4   D(N1+LL)=1.(N1+LL-1)+D4
5   IF(IE.EQ.0)GO TO 30
    IF(IE.EQ.-1)GO TO 25
    NS=N-1
    DO 20 L=1,N
      IF(L.NE.IE)GO TO 20
      DO 15 M=L,NS
15   D(M)=D(M+1)
20   CONTINUE
      GO TO 30
25   K=0
    K1=1
    DO 26 K2=1,N
      IF(IND(K2).EQ.1)GO TO 26
      AD(K1)=D(IP(K2))
      K=K+1
      K1=K1+1
26   CONTINUE
    DO 28 K3=1,K
      D(K3)=AD(K3)
      JP(K3)=K3
    DO 27 K4=1,K
      IF(D(K3).LE.AD(K4))GO TO 27
      D(K3)=AD(K4)

```

```

      JP(K3)=K4
27  CONTINUE
      AD(JP(K3))=1.0E 6
28  CONTINUE
      NS=K
30  DO 60 I=1,NS
      DO 60 J=I,NS
      X=(D(J)-D(I))*2.0D0*PI
      IF(I-J)40,50,40
40  CONTINUE
      A(I,J)=(1.0D0-CDEXP(DCMPLX(0.D0,X*KN)))
      A(I,J)=A(I,J)/(1.D0-CDEXP(DCMPLX(0.D0,X)))
      A(J,I)=DCONJG(A(I,J))
      GO TO 60
50  A(I,J)=1.0D0*KN
60  CONTINUE
      DO 80 K=1,NS
      COL(K)=(0.D0,0.D0)
      Y3=-2.D0*PI*D(K)
      DO 80 J=1,KN
      COL(K)=COL(K)+Y(J)*CDEXP(DCMPLX(0.D0,Y3*(J-1)))
80  CONTINUE
      RETURN
      END
C
C
C      SUBROUTINE AHERIV IS FOR SOLVING THE LINEAR
C      EQUATIONS AC=B AND CORRESPONDING ERROR FUNCTION
C
      SUBROUTINE AHERIV(A,COL,SOL,Y,ET,K,KN)
      COMMON PI,A1,A2,B1,B2,D1,D2,D3,D4,N1,N2
      REAL*8 A1,A2,B1,B2,PI,C,VN(50),C1,C2
      COMPLEX*16 A(50,50),COL(50),RL(50,50),SOL(50)
      COMPLEX*16 XX1,Y(100),E1,E2,E3,ET,XX
      DO 5 I=1,K
      DO 5 J=1,K
5       RL(I,J)=DCMPLX(0.0D0,0.0D0)
      DO 60 I=1,K
      IF1=I+1
      IM1=I-1
      IF(I.NE.1)GO TO 30
      DO 10 J=1,K
10     RL(1,J)=A(1,J)
      DO 20 J=2,K
20     RL(J,1)=DCONJG(RL(1,J))/RL(1,1)
      GO TO 60
30     DO 40 J=I,K
      RL(I,J)=A(I,J)
      DO 40 M=1,IM1
40     RL(I,J)=RL(I,J)-RL(I,M)*RL(M,J)

```

```

      IF(I.EQ.K)GO TO 60
      DO 45 J=IP1,K
45   RL(J,I)=DCONJG(RL(I,J))/RL(I,I)
60   CONTINUE
80   DO 100 I=1,K
     IM1=I-1
     SOL(I)=COL(I)
     IF(I.EQ.1)GO TO 100
     DO 90 L=1,IM1
90   SOL(I)=SOL(I)-SOL(L)*RL(I,L)
100  CONTINUE
     DO 140 I=1,K
        J=K-I+1
        IF(J.EQ.K)GO TO 130
        JP1=J+1
        DO 120 L=JP1,K
120  SOL(J)=SOL(J)-SOL(L)*RL(J,L)
130  SOL(J)=SOL(J)/RL(J,J)
140  CONTINUE
C
C      EVALUATION OF ERROR, EQUATION (4.2.19)
C
      E2=0.0D0
      E1=0.0D0
      DO 15 I1=1,K
      E2=E2+SOL(I1)*DCONJG(COL(I1))
15   CONTINUE
      DO 16 I3=1,KN
      E1=E1+CDARS(Y(I3))**2
16   CONTINUE
      ET=(E1-E2)/KN
      RETURN
      ENII
C
C      SUBROUTINE COMBO IS FOR GENERATING STRING
C      OF ONES AND ZEROS FOR DELETION OF PROPER
C      BASIS FUNCTIONS
C
      SUBROUTINE COMBO(JN,N,M,NF)
      DIMENSION N(40)
      IONES=M
      NF=0
1     NF=1
2     CONTINUE
      IF(JN(NF).EQ.0)GO TO 10
      IF(NF.EQ.N) 0 TO 101
      JN(NF)=0
      NF=NF+1
      IONES=IONES-1

```

```
      GO TO 2
10  JN(NP)=1
    IONES=IONES+1
    IF(IONES.EQ.M)GO TO 100
    GO TO 1
101 NF=1
100 RETURN
END

C
C
C      SUBROUTINE GAUSS AND RANDU ARE FOR
C      GENERATING GAUSSIAN NOISE
C
C      SUBROUTINE GAUSS(IX,S,AM,V)
A=0.0
DO 50 I=1,12
CALL RANDU(IX,IY,Y)
IX=IY
50 A=A+Y
V=(A-6.0)*S+AM
RETURN
END
SUBROUTINE RANDU(IX,IY,YFL)
IY=IX*65539
IF(IY)5,6,6
5  IY=IY+2147483647+1
6  YFL=IY
YFL=YFL*.4656613E-9
RETURN
END

C
C
C      SUBROUTINE DELT IS FOR DELETING PROPER ROW
C      AND COLUMN OF A AND B MATRICES FOR
C      EVALUATION OF EI ERROR FUNCTIONS
C
C      SUBROUTINE DELT(A,COL,S,RCOL,N,NS,IE)
COMPLEX*16 A(50,50),COL(50),S(50,50),RCOL(50)
IEM=IE-1
IF(IE.EQ.1)GO TO 30
DO 10 I=1,IEM
DCOL(I)=COL(I)
DO 10 J=I,IEM
S(I,J)=A(I,J)
S(J,I)=A(J,I)
10  CONTINUE
DO 20 I=1,IEM
DO 20 J=IE,NS
S(I,J)=A(I,J+1)
S(J,I)=A(J+1,I)
```

```
20  CONTINUE
IF(IE.EQ.N)GO TO 50
30  DO 40 I=IE,NS
DCOL(I)=COL(I+1)
DO 40 J=I,NS
S(I,J)=A(I+1,J+1)
S(J,I)=A(J+1,I+1)
40  CONTINUE
50  RETURN
END
```

```

C      PROGRAM-4
C
C
C      THIS IS THE PROGRAM LISTING FOR TWO
C      FREQUENCY ESTIMATION BY COMPLEX SUBSET
C      SELECTION AND WINDOWING TECHNIQUE.
C
C      A1,A2      TONES AMPLITUDES
C      B1,B2      TONES FREQUENCIES
C      KN         NO. OF SAMPLES
C      A(I,J)    A MATRIX
C      COL(I)    B MATRIX
C      SOL(I)    COEFFICIENT MATRIX
C      E(I)      EI ERROR FUNCTIONS
C      SIR       SIGNAL TO INTERFERENCE RATIO
C      SNR       SIGNAL TO NOISE RATIO
C      NRUN      NO. OF RUNS
C
COMMON PI,A1,A2,R1,R2,D1,D2,D3,D4,N1,N2,KN
REAL*8 D(40),E(40),AE(40),A1,A2,B1,B2,PI
REAL*8 D1,D2,D3,D4,RR,ETA,ETAN,P1,P2
REAL*8 ED11(100),ED22(100),DMER1,DMER2,STD1,STD2
REAL*8 ED1(100),ED2(100),RMEAN1,RMEAN2,VAR1,VAR2
INTEGER IP(40),IND(40),INDB(40),INP(40),JP(40)
INTEGER LJP(40),R,JJP(40)
COMPLEX*16 A(40,40),COL(40),SOL(40),SOL1(40),ET
COMPLEX*16 VN,S(40,40),DCOL(40),BRSS,Y(100)
DATA A,COL/1600*(0.000,0.000),40*(0.00,0.00)/
DATA IND/40*0/
PI=4.000*DATAN(1.000)
READ(5,110)NRUN,SIR,SNR,B1,B2
READ(5,105)KN,N1,MS,ML
A1=1.00
A2=A1*(10**(-SIR/20))
WRITE(6,230)KN,NRUN
WRITE(6,350)A1,A2
WRITE(6,360)B1,B2
KPT=0
IX=3657137
SD=A1*(10**(-SNR/20))
VAR=SD**2
WRITE(6,280)SIR,SNR,VAR
P1=2.00*PI*B1
P2=2.00*PI*B2
C
C      BEGINING OF EACH RUN
C
DO 500 IJ=1,NRUN
C

```

```

C      GENERATION OF GAUSSIAN NOISE IN DATA,
C      EQUATIONS (4.2.2) AND (4.4.1)
C
C      DO 1 I=1,KN
C      CALL GAUSS(IX,SD,O.,V)
C      CALL GAUSS(IX,SD,O.,V1)
C      VN=CMPLX(V,V1)
C      Y(I)=A1*CDEXP(DCMPLX(O,DO,P1*(I-1)))
C      Y(I)=Y(I)+A2*CDEXP(DCMPLX(O,DO,P2*(I-1)))+VN
1    CONTINUE
C
C      GENERATION OF BASIS FUNCTION PARAMETERS
C
C      N=10
C      NO=0
C      LFI=0
C      D1=.1
C      D2=D1
C      D(1)=0.00
C      DO 2 I=2,10
2    D(I)=D(I-1)+D1
C      NS=N
C      GO TO 7
3    LFI=1
C      D(1)=D3
C      D(6)=D4
C      DO 4 J=2,5
4    D(J)=D(J-1)+D1
C      D(J+5)=D(J+4)+D1
C      N=10
C      NS=N
7    IE=0
C      CALL INTG(A,COL,Y,IND,IP,N,NS,LFI,IE)
C      CALL AHERIV(A,COL,SOL,Y,ET,NS,KN)
C      ETA=CDABS(ET)
C
C      EVALUATION AND RE-ORDERING OF EI ERROR
C      FUNCTIONS, EQUATIONS (2.2.9) AND (2.2.10)
C
C      NS=N-1
C      DO 40 IE=1,N
C      CALL DELT(A,COL,S,DCOL,N,NS,IE)
C      CALL AHERIV(S,DCOL,SOL,Y,ET,NS,KN)
C      E(IE)=CDABS(ET)
40   CONTINUE
C      DO 60 K=1,N
C      AE(K)=E(K)
C      IP(K)=K
C      DO 50 LP=1,N
C      IF(AE(K).LE.E(LP))GO TO 50

```

```

AE(K)=E(LP)
IP(K)=LP
50 CONTINUE
E(IP(K))=1.0E 6
60 CONTINUE
C
C BEGINING OF THE SEARCH FOR BEST SUBSET
C
DO 99 M=MS,ML
NR=N-M
DO 70 I=1,NR
IND(I)=1
INDB(I)=1
70 CONTINUE
DO 80 J=1,M
IND(J+NR)=0
INDB(J+NR)=0
80 CONTINUE
IE=-1
MP=NR+1
BR=AE(MP)
KP=1
KPT=KPT+1
MI=M+1
ETAN=1.0E 6
DO 90 IT=1,MI
IF(IT.EQ.1)GO TO 82
KP=(KP*(NR+IT-2))/(IT-1)
KPT=KPT+KP
82 DO 86 K=1,KP
IF(IT.EQ.1)GO TO 83
CALL COMBO(IND,N,NR,NF)
83 CALL INTG(A,COL,Y,IND,IP,N,NS,LFI,IE)
CALL AHERIV(A,COL,SOL,Y,ET,NS,KN)
ETA=CDABS(ET)
IF(ETA.GT.ETAN)GO TO 86
ETAN=ETA
DO 84 I=1,N
INDB(I)=IND(I)
84 CONTINUE
DO 85 J=1,M
SOL1(J)=SOL(J)
85 CONTINUE
86 CONTINUE
87 IF(ETAN.LE.BR)GO TO 92
88 MP=MP+1
BR=AE(MP)
90 CONTINUE
92 J=0
DO 93 L=1,N

```

```

        IF(INDB(L).EQ.1)GO TO 93
        J=J+1
        JP(J)=IP(L)
93    CONTINUE
        DO 95 LP=1,J
        LJP(LP)=JP(LP)
        JJP(LP)=LP
        DO 94 KP=1,J
        IF(LJP(LP).LE.JP(KP))GO TO 94
        LJP(LP)=JP(KP)
        JJP(LP)=KP
94    CONTINUE
        JP(JJP(LP))=1E 6
95    CONTINUE
C
C      APPLYING WINDOWS AROUND SELECTED
C      BASIS FUNCTIONS
C
        NO=NO+1
        D1=D1/2
        D2=D1
        IF(NO.GE.2)GO TO 505
        D3=D(LJP(1))-D1
        D4=D(LJP(2))-D1
        GO TO 506
505  MS=2
        ML=2
        IF(DABS(D(LJP(1))-D(LJP(2))).LE..002D0)GO TO 550
        D3=D(LJP(1))-2*D1
        D4=D(LJP(2))-2*D1
506  IF(D3.GE.0.D0)GO TO 520
        D3=0.D0
        IF(D4-D3)525,525,520
525  D4=D4+2*D1
520  IF(D4-D3-4*D1)501,502,503
501  D4=D4+3*D1
        GO TO 503
502  D4=D4+D1
503  IF(NO.GE.7)GO TO 98
        IF(ETAN.GT.1.0E-10)GO TO 3
98    CONTINUE
99    CONTINUE
550  ED1(IJ)=D(LJP(1))
        ED2(IJ)=D(LJP(2))
        WRITE(6,300)ED1(IJ),ED2(IJ)
500  CONTINUE
C
C      EVALUATION OF MEAN AND VARIANCE OF
C      THE ESTIMATES
C

```

AD-A090 128      RHODE ISLAND UNIV KINGSTON DEPT OF ELECTRICAL ENGIN--ETC F/G 9/2  
APPLICATIONS OF EFFICIENT SUBSET SELECTION TO DIGITAL FILTERING--ETC(U)  
AUG 80 J KHOARAMMI, D TUFTS AFOSR-77-3174

UNCLASSIFIED

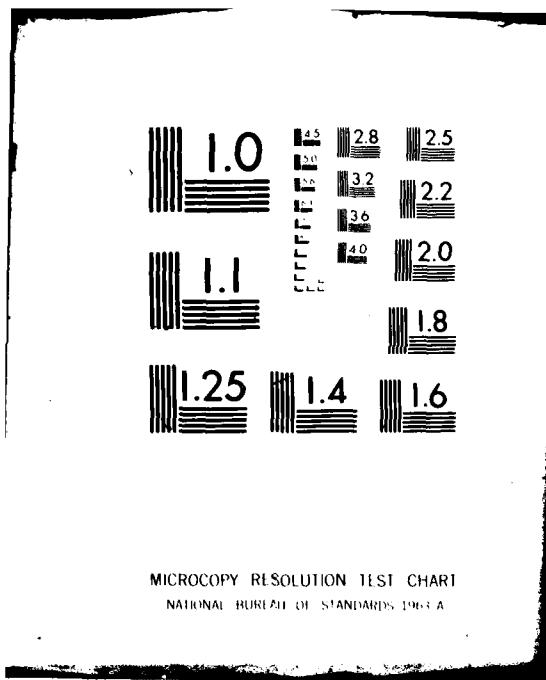
2 OF 2

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```

CALL VARIAN(ED1,ED2,DMEAN1,DMEAN2,VAR1,VAR2,NRUN)
WRITE(6,290)
DO 510 III=1,NRUN
  ED11(III)=DABS(ED1(III)-B1)
  ED22(III)=DABS(ED2(III)-B2)
  WRITE(6,300)ED1(III),ED11(III),ED2(III),ED22(III)
510 CONTINUE
DMER1=DABS(B1-DMEAN1)
DMER2=DABS(B2-DMEAN2)
STD1=DSQRT(VAR1)
STD2=DSQRT(VAR2)
WRITE(6,310)DMEAN1,DMEAN2
WRITE(6,330)DMER1,DMER2
WRITE(6,320)VAR1,VAR2
WRITE(6,340)STD1,STD2
100 FORMAT(I1,3I2,2F10.0)
105 FORMAT(2I3,2I2)
110 FORMAT(I3,4F10.0)
120 FORMAT(5X,'D('',I2,'')='',E14.6)
130 FORMAT(/,5X,'C('',I2,'')=''',E12.4,' ',',
*E12.4,' '')')
140 FORMAT(/,5X,'RSS FOR FULL BASIS ='',E12.4)
150 FORMAT(/,5X,'N='',I2,' M='',I2)
160 FORMAT(/,5X,'E('',I2,'')='',E12.4)
170 FORMAT(/,5X,'E('',I2,'')='',E12.4,' NO. ',I2
*, ' SMALLEST ERROR')
180 FORMAT(/,5X,'C('',I2,'')=''',E12.4,' ',',E12.4,' '')')
190 FORMAT(/,5X,'ERROR='',E12.4)
200 FORMAT(/,5X,'NO. OF SUBSETS EXAMINED = ',I4)
210 FORMAT(/,5X,'N='',I2)
220 FORMAT(2I2,2F10.0)
230 FORMAT(/,5X,'KN='',I3,3X,'NRUN='',I3)
280 FORMAT(/,5X,'SIR = ',E10.4,3X,'SNR = ',E10.4,3X
*, 'NVAR='',E10.4)
290 FORMAT(/,9X,'FREQ1',7X,'FREQ-ERROR1',7X,'FREQ2'
*,7X,'FREQ-ERROR2')
300 FORMAT(/,2X,4(3X,E12.6))
310 FORMAT(/,5X,'MEAN1 = ',E12.6,7X,'MEAN2 = ',E12.6)
320 FORMAT(/,5X,'VARIANCE1='',E12.6,5X,'VARIANCE2='',E12.6)
330 FORMAT(/,5X,'MERROR1='',E12.6,7X,'MERROR2='',E12.6)
340 FORMAT(/,5X,'STDIV1 = ',E12.6,5X,'STDIV2 = ',E12.6)
350 FORMAT(/,5X,'A1='',E10.4,3X,'A2='',E10.4)
360 FORMAT(/,5X,'F1='',E10.4,3X,'F2='',E10.4)
STOP
END
C
C
C   SUBROUTINE INTG IS FOR EVALUATION OF ELEMENTS OF
C   A AND R MATRICES FOR ACH SUBSET TO BE EXAMINED,
C   EQUATIONS (4.2.14) THRU (4.2.16)

```

C

```

SUBROUTINE INTG(A,COL,Y,IND,IP,N,NS,LFI,IE)
COMMON PI,A1,A2,B1,B2,D1,D2,D3,D4,N1,N2,KN
COMPLEX*16 A(40,40),COL(40),Y(100)
REAL*8 D(40),AD(40),X,Y1,Y2,Y3,A1,A2,B1,B2,PI
REAL*8 D1,D2,D3,D4
INTEGER IND(40),IP(40),JP(40)
IF(LFI.EQ.1)GO TO 2
D(1)=D1
DO 1 L=2,N
1 D(L)=D(L-1)+D1
GO TO 5
2 D(1)=D3
D(6)=D4
DO 3 L=2,5
D(L)=D(L-1)+D1
3 D(L+5)=D(L+4)+D1
5 IF(IE.EQ.0)GO TO 30
IF(IE.EQ.-1)GO TO 25
NS=N-1
DO 20 L=1,N
IF(L.NE.IE)GO TO 20
DO 15 M=L,NS
15 D(M)=D(M+1)
20 CONTINUE
GO TO 30
25 K=0
K1=1
DO 26 K2=1,N
IF(IND(K2).EQ.1)GO TO 26
AD(K1)=D(IP(K2))
K=K+1
K1=K1+1
26 CONTINUE
DO 28 K3=1,K
D(K3)=AD(K3)
JP(K3)=K3
DO 27 K4=1,K
IF(D(K3).LE.AD(K4))GO TO 27
D(K3)=AD(K4)
JP(K3)=K4
27 CONTINUE
AD(JP(K3))=1.0E 6
28 CONTINUE
NS=K
30 DO 60 I=1,NS
DO 60 J=I,NS
X=(D(J)-D(I))*2.0D0*PI
IF(X)40,50,40
40 CONTINUE

```

```

A(I,J)=(1.0D0-CDEXP(DCMPLX(0,D0,X*KN)))
A(I,J)=A(I,J)/(1.D0-CDEXP(DCMPLX(0,D0,X)))
A(J,I)=DCONJG(A(I,J))
GO TO 60
50 A(I,J)=1.0D0*KN+.00001
60 CONTINUE
DO 80 K=1,NS
COL(K)=(0.D0,0.D0)
Y3=-2.D0*PI*D(K)
DO 80 J=1,KN
COL(K)=COL(K)+Y(J)*CDEXP(DCMPLX(0,D0,Y3*(J-1)))
80 CONTINUE
RETURN
END

C
C
C SUBROUTINE AHERIV IS FOR SOLVING THE LINEAR
C EQUATIONS AC=B AND CORRESPONDING ERROR FUNCTION
C
SUBROUTINE AHERIV(A,COL,SOL,Y,ET,K,KN)
COMMON PI,A1,A2,B1,B2,D1,D2,D3,D4,N1,N2
REAL*8 A1,A2,B1,B2,PI,C,VN(50),C1,C2
COMPLEX*16 A(40,40),COL(40),RL(40,40),SOL(40)
COMPLEX*16 XX1,Y(100),E1,E2,E3,ET,XX
DO 5 I=1,K
DO 5 J=1,K
5 RL(I,J)=DCMPLX(0.0D0,0.0D0)
DO 60 I=1,K
IP1=I+1
IM1=I-1
IF(I.NE.1)GO TO 30
DO 10 J=1,K
10 RL(1,J)=A(1,J)
DO 20 J=2,K
20 RL(J,1)=DCONJG(RL(1,J))/RL(1,1)
GO TO 60
30 DO 40 J=I,K
RL(I,J)=A(I,J)
DO 40 M=1,IM1
40 RL(I,J)=RL(I,J)-RL(I,M)*RL(M,J)
IF(I.EQ.K)GO TO 60
DO 45 J=IP1,K
45 RL(J,I)=DCONJG(RL(I,J))/RL(I,I)
60 CONTINUE
80 DO 100 I=1,K
IM1=I-1
SOL(I)=COL(I)
IF(I.EQ.1)GO TO 100
DO 90 L=1,IM1
90 SOL(I)=SOL(I)-SOL(L)*RL(I,L)

```

```

100 CONTINUE
    DO 140 I=1,K
    J=K-I+1
    IF(J.EQ.K)GO TO 130
    JP1=J+1
    DO 120 L=JP1,K
120  SOL(J)=SOL(J)-SOL(L)*RL(J,L)
130  SOL(J)=SOL(J)/RL(J,J)
140 CONTINUE

C
C      EVALUATION OF ERROR, EQUATION (4.2.19)
C
        E2=0.0D0
        E1=0.0D0
        DO 15 I1=1,K
        E2=E2+SOL(I1)*DCONJG(COL(I1))
15   CONTINUE
        DO 16 I3=1,KN
        E1=E1+CDARS(Y(I3))**2
16   CONTINUE
        ET=E1-E2
        RETURN
        END

C
C      SUBROUTINE COMBO IS FOR GENERATING STRING
C      OF ONES AND ZEROS FOR PROPER DELETION OF
C      BASIS FUNCTIONS
C
        SUBROUTINE COMBO(JN,N,M,NF)
        DIMENSION JN(40)
        IONES=M
        NF=0
1      NP=1
2      CONTINUE
        IF(JN(NP).EQ.0)GO TO 10
        IF(NP.EQ.N)GO TO 101
        JN(NP)=0
        NP=NP+1
        IONES=IONES-1
        GO TO 2
10     JN(NP)=1
        IONES=IONES+1
        IF(IONES.EQ.M)GO TO 100
        GO TO 1
101    NF=1
100   RETURN
        END

C
C

```

```
C      SUBROUTINE GAUSS AND RANDU ARE FOR
C      GENERATING GAUSSIAN NOISE
C
C      SUBROUTINE GAUSS(IX,S,AM,V)
A=0.0
DO 50 I=1,12
CALL RANDU(IX,IY,Y)
IX=IY
50 A=A+Y
V=(A-6.0)*S+AM
RETURN
END
SUBROUTINE RANDU(IX,IY,YFL)
IY=IX*65539
IF(IY)5,6,6
5 IY=IY+2147483647+1
6 YFL=IY
YFL=YFL*.4656613E-9
RETURN
END
C
C      SUBROUTINE VARIAN IS FOR EVALUATING MEAN
C      AND VARIANCE OF THE FREQUENCY ESTIMATES
C
SUBROUTINE VARIAN(ED1,ED2,DMEAN1,DMEAN2,VAR1,VAR2,NRUN)
REAL*8 ED1(100),ED2(100),DMEAN1,DMEAN2,VAR1
REAL*8 VAR2,X1,X2
VAR1=0.D0
VAR2=0.D0
DMEAN1=0.D0
DMEAN2=0.D0
DO 10 I=1,NRUN
DMEAN1=DMEAN1+ED1(I)
DMEAN2=DMEAN2+ED2(I)
10 CONTINUE
DMEAN1=DMEAN1/NRUN
DMEAN2=DMEAN2/NRUN
DO 20 J=1,NRUN
VAR1=VAR1+(ED1(J)-DMEAN1)**2
VAR2=VAR2+(ED2(J)-DMEAN2)**2
20 CONTINUE
VAR1=VAR1/NRUN
VAR2=VAR2/NRUN
RETURN
END
C
C      SUBROUTINE DELT IS FOR DELETING PROPER
C      ROW AND COLUMN OF A AND B MATRICES FOR
```

C EVALUATION OF EI ERROR FUNCTIONS  
C

```
SUBROUTINE DELT(A, COL, S, DCOL, N, NS, IE)
COMPLEX*16 A(40,40),COL(40),S(40,40),DCOL(40)
IEM=IE-1
IF(IE.EQ.1)GO TO 30
DO 10 I=1,IEM
DCOL(I)=COL(I)
DO 10 J=I,IEM
S(I,J)=A(I,J)
S(J,I)=A(J,I)
10 CONTINUE
DO 20 I=1,IEM
DO 20 J=IE,NS
S(I,J)=A(I,J+1)
S(J,I)=A(J+1,I)
20 CONTINUE
IF(IE.EQ.N)GO TO 50
30 DO 40 I=IE,NS
DCOL(I)=COL(I+1)
DO 40 J=I,NS
S(I,J)=A(I+1,J+1)
S(J,I)=A(J+1,I+1)
40 CONTINUE
50 RETURN
END
```

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